

Assignment 2

Due Monday, 7/7/2014

Remember to show work for credit!

Problem 1

Solve each of the following initial value problems.

(a) $y' + 4y = x^2e^{-4x}$, $y(0) = -2$.

(b) $y' + 2y = x^2e^{-4x}$, $y(0) = -2$.

(c) $y' = \frac{2x}{y^2+x^2y^2}$, $y(0) = 1$.

(d) $t \cdot \frac{dy}{dt} + 2y = \sin(t)$, $y(\pi) = 1$.

(e) $\frac{dy}{dx} = 2x + 4xy$, $y(0) = 3$.

Problem 2

Decide whether each of the following equations is exact. Solve each differential equation if possible. Implicit solutions are fine.

(a) $(3 - 2y^2 + 4xy)y' = 2y^2 - 3x + 1$.

(b) $\frac{dy}{dx} = \frac{x-y}{x+5y^{-1}}$.

(c) $\frac{dy}{dt} = \frac{y-t^3}{t+y^3}$.

Problem 3

Solve the following differential equation. An implicit solution is fine. (Hint: try the substitution $v = y/x$.)

$$y' = (x - y) / (x + y).$$

Problem 4 – Tumor Growth

In class we looked at two models of population growth: exponential and logistic. In exponential growth, the per capita growth rate was a constant, r . That is,

$$\frac{1}{N} \frac{dN}{dt} = r.$$

In logistic growth, the per capita growth rate was a linear, decreasing function where the growth rate was r for very low densities and dropped to zero when the population reached K . That is,

$$\frac{1}{N} \frac{dN}{dt} = r \cdot \left(1 - \frac{N}{K}\right).$$

In this problem, we will suppose that the per capita growth rate is unbounded for very low densities and drops off to zero when the population reaches K . In particular, suppose

$$\frac{1}{N} \frac{dN}{dt} = r \ln \left(\frac{K}{N}\right),$$

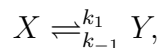
where $r, K > 0$ are parameters and $N(t)$ represents the number of cancerous cells in an organ. (This is called the Gompertz equation, and it has been used to model the growth of tumors.)

- Sketch the graph of $\frac{dN}{dt}$ vs. N , find the critical points and determine whether each is stable or unstable.
- Solve this equation using the initial value $N(0) = N_0$, where $N_0 > 0$ is a constant. Does this solution agree with your results from (a)? In particular, what value does N approach as $t \rightarrow \infty$?
- Sketch $N(t)$ vs. t for various initial conditions. Choose at least one initial condition $N_0 < K$, one $N_0 = K$ and at least one $N_0 > K$.

Problem 5 – Chemical Reactions

One of the simplest (at least to model) types of chemical reactions is called an isomerization. This is a reaction where one molecule changes into a different molecule with the same atoms, but a different structure. For example, the molecule cyclohexane converts between two (rather whimsically named) structures: the chair conformation

and the twist-boat conformation. If we call chair-conformation cyclohexane X and twist-boat conformation cyclohexane Y , this can be written as



Where $k_1, k_{-1} > 0$ are the forward and backward reaction rates, respectively.

- (a) If the concentration of Y is extremely low relative to X (perhaps because we are removing twist-boat cyclohexanes as soon as they form), then we can ignore the backwards reaction. The concentration of chair conformation cyclohexane (denoted by $x(t)$), can then be modeled as

$$\frac{dx}{dt} = -k_1x.$$

Solve for the concentration $x(t)$, assuming the initial concentration is $x(0) = x_0$. Sketch the solution curve.

- (b) If the concentration of Y is not extremely low, but is somehow held constant at some concentration y_0 , then the concentration of x can be modeled as

$$\frac{dx}{dt} = k_{-1}y_0 - k_1x.$$

(This model is also valid if the concentration of Y is extremely high relative to x .) Solve for the concentration $x(t)$, assuming the initial concentration is $x(0)$. Sketch the solution curve.