Assignment 2

Due Monday, 7/7/2014

Remember to show work for credit!

Problem 1

Solve each of the following initial value problems.

(a) $y' + 4y = x^2 e^{-4x}$, $y(0) = -2$.

(b) $y' + 2y = x^2 e^{-4x}$, $y(0) = -2$.

(c) $y' = \frac{2x}{y^2 + x^2 y^2}$, $y(0) = 1$.

(d) $t \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \sin(t)$, $y(\pi) = 1$.

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 4xy$, $y(0) = 3$.

Problem 2

Decide whether each of the following equations is exact. Solve each differential equation if possible. Implicit solutions are fine.

(a) $(3 - 2y^2 + 4xy)y' = 2y^2 - 3x + 1$.

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+5y^{-1}}$.

(c) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y-t^3}{t+y^3}$.

Problem 3

Solve the following differential equation. An implicit solution is fine. (Hint: try the substitution $v = y/x$.)

$$y' = (x - y) \,/\, (x + y).$$

Problem 4 – Tumor Growth

In class we looked at two models of population growth: exponential and logistic. In exponential growth, the per capita growth rate was a constant, r. That is,

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = r.$$

In logistic growth, the per capita growth rate was a linear, decreasing function where the growth rate was r for very low densities and dropped to zero when the population reached K. That is,

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = r \cdot \left(1 - \frac{N}{K}\right).$$

In this problem, we will suppose that the per capita growth rate is unbounded for very low densities and drops off to zero when the population reaches K. In particular, suppose

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = r \ln\left(\frac{K}{N}\right),$$

where $r, K > 0$ are parameters and $N(t)$ represents the number of cancerous cells in an organ. (This is called the Gompertz equation, and it has been used to model the growth of tumors.)

(a) Sketch the graph of $\frac{\mathrm{d}N}{\mathrm{d}t}$ vs. N, find the critical points and determine whether each is stable or unstable.

(b) Solve this equation using the initial value $N(0) = N_0$, where $N_0 > 0$ is a constant. Does this solution agree with your results from (a)? In particular, what value does N approach as $t \to \infty$?

(c) Sketch $N(t)$ vs. t for various initial conditions. Choose at least one initial condition $N_0 < K$, one $N_0 = K$ and at least one $N_0 > K$.

Problem 5 – Chemical Reactions

One of the simplest (at least to model) types of chemical reactions is called an isomerization. This is a reaction where one molecule changes into a different molecule with the same atoms, but a different structure. For example, the molecule cyclohexane converts between two (rather whimsically named) structures: the chair conformation and the twist-boat conformation. If we call chair-conformation cyclohexane X and twist-boat conformation cyclohexane Y, this can be written as

$$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} Y,$$

Where $k_1, k_{-1} > 0$ are the forward and backward reaction rates, respectively.

(a) If the concentration of Y is extremely low relative to X (perhaps because we are removing twist-boat cyclohexanes as soon as they form), then we can ignore the backwards reaction. The concentration of chair conformation cyclohexane (denoted by $x(t)$), can then be modeled as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k_1 x.$$

Solve for the concentration $x(t)$, assuming the initial concentration is $x(0) = x_0$. Sketch the solution curve.

(b) If the concentration of Y is not extremely low, but is somehow held constant at some concentration y_0, then the concentration of x can be modeled as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k_{-1}y_0 - k_1 x.$$

(This model is also valid if the concentration of Y is extremely high relative to x.) Solve for the concentration $x(t)$, assuming the initial concentration is $x(0)$. Sketch the solution curve.