Logistic Model with Harvesting

Model

Here, we will talk about a model for the growth and harvesting of a fish population. Suppose that, absent any fishermen, the population grows logistically. That is,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right),$$

where $N(t)$ is the abundance of fish, $K > 0$ is the maximum population that the environment can support (called the carrying capacity) and $r > 0$ is the maximum growth rate. Remember that these are all, at least in principle, measurable parameters. If we let the population grow for a while undisturbed, it should approach some constant abundance K. If we look at a very small population (say by isolating a few fish), we can measure r by counting how many are left at the end of one year. The units for these values are as follows:

$$\begin{aligned} N &= \#\text{ of fish (unitless)}, \\ K &= \#\text{ of fish (unitless)}, \\ r &= \#\text{ of fish/time} = \text{time}^{-1}. \end{aligned}$$

Now suppose that people exert a constant amount of effort harvesting fish. This means that they will reduce the growth rate of the fish by some amount. In particular, we will assume this reduction in growth rate is of the form $-qEN$. Here E is the amount of effort exerted per time. (The units for effort end up not mattering. One reasonable choice is total fishing hours.) The parameter q is called the catchability coefficient. It is the fraction of the total population caught for every unit of effort exerted. In our model, we assume that both q and E are constant. That is, they depend on neither time nor fish density.

The units for our parameters are therefore

$$\begin{aligned} q &= \text{fraction/effort}, \\ E &= \text{effort/time}. \end{aligned}$$

This means that qEN has units of # of fish/*time*, just like the other terms in the model. To simplify matters, we will define $h \equiv qE$.

Our final equation is therefore

$$\frac{\mathrm{d}N}{\mathrm{d}t} = f(N) = rN\left(1 - \frac{N}{K}\right) - hN. \tag{1}$$

This model is separable, so we can actually solve it explicitly. (It is also a Bernoulli equation. We won't worry about this too much, but it means that the substitution $y = N^{-1}$ would make the problem linear.) However, most nonlinear problems are not so easy to solve, so let's try to analyze this qualitatively (that is, without finding a formula for the solution).

Equilibria

The first thing we need to do is find the equilibria of our system. We do this by setting the derivative equal to zero. This gives

$$rN\left(1 - \frac{N}{K}\right) - hN = 0,$$

so

$$rN\left(1 - \frac{N}{K} - \frac{h}{r}\right) = 0,$$

so either

$$rN = 0, \quad \text{or} \quad 1 - \frac{N}{K} - \frac{h}{r} = 0.$$

This means our equilibria (or fixed points) are

$$N^* = 0 \quad \text{and} \quad N^* = K \cdot \left(1 - \frac{h}{r}\right).$$

This should make a lot of sense. Any reasonable model of population growth should have one equilibrium at 0. This means that if the population is extinct it stays extinct; no one is adding fish from outside. The second equilibrium is less than K (remember that both r and h are necessarily positive). The population would reach an equilibrium at K if there were no harvesting, so it seems natural that harvesting would reduce the equilibrium below K.

Graphical Approach

Before we do any more calculations, let's see what we can learn from graphing these functions. First, let's graph $\frac{\mathrm{d}N}{\mathrm{d}t}$ as a function of N.

For light fishing (small h), there are two equilibria: one at 0 and another below K, just as expected. For heavier fishing (higher h), there are still two equilibria, but the nonzero equilibrium has become negative.

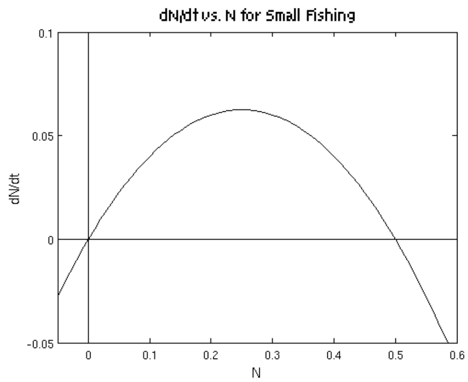

Figure 1: $\frac{\mathrm{d}N}{\mathrm{d}t}$ for light fishing. Here we have chosen $r = 1$, $K = 1$ and $h = 0.5$.

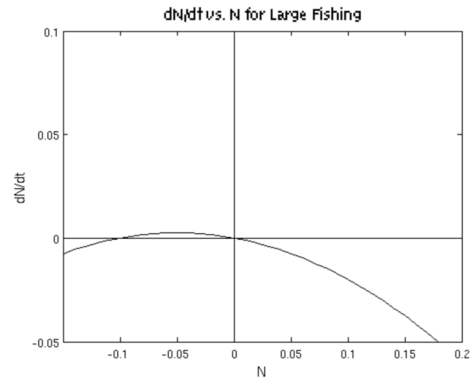

Figure 2: $\frac{\mathrm{d}N}{\mathrm{d}t}$ for overfishing. Here, $h = 1.1$.

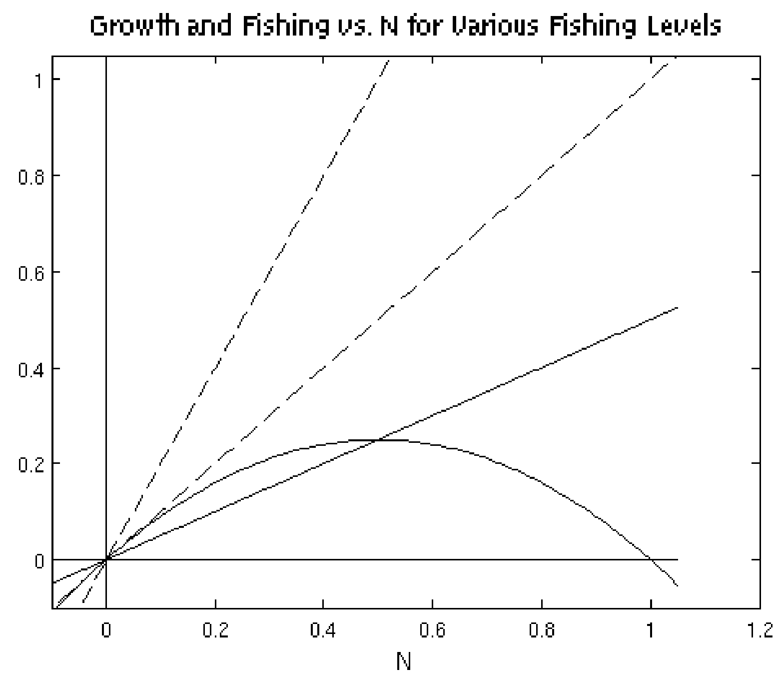

Figure 3: Here is a plot of the growth terms (the parabola) and fishing terms (the lines) for various values of h. Equilibria ocurr when the two curves cross. As h increases, the point of intersection moves closer and closer to zero. For sufficiently large h, the positive equilibrium disappears entirely.

Another useful way to look at this problem is to plot each term from $\frac{\mathrm{d}N}{\mathrm{d}t}$. Fixed points ocurr when the growth term $rN\left(1 - N/K\right)$ crosses the harvesting term hN. As h increases (which corresponds to the effort put into fishing), the slope of the line increases, which means that the positive equilibrium moves closer to 0. When the slope reaches $h = r$, the positive equilibrium disappears. When h increases even more, a negative equilibrium appears.

Stability

We can check the stability of each fixed poit graphically, but you should only trust graphs so far! To determine stability analytically, we need to linearize about each fixed point. First, let's look at $N^* = 0$. We want to know what happens to populations that start very close to 0. For very small N, notice that N^2 is much smaller than N. Since $\frac{\mathrm{d}N}{\mathrm{d}t}$ is a polynomial, we can linearize simply by throwing away all terms with higher powers. This means,

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= rN\left(1 - \frac{N}{K}\right) - hN \\ &= rN - \frac{rN^2}{K} - hN \\ &\approx rN - hN \\ &= (r-h)N.\end{aligned}$$

This is an example of the most important differential equation we have studied:

$$y' = \lambda y.$$

Remember that this equation has a stable equilibrium at 0 when $\lambda < 0$ and an unstable equilibrium at 0 when $\lambda > 0$. (When $\lambda = 0$, our life gets substantially more complicated, but we won't worry about that here.)

In our case, $\lambda = r - h$. When $r > h$ this is positive, so $N^* = 0$ is unstable. When $r < h$ this is negative, so $N^* = 0$ is stable. This should make a lot of sense. When fishing effort is small compared to the fish growth rate, the population will persist. When fishing effort is too big (bigger than the growth rate), the fish will go extinct.

Notice that we could also have achived this result by looking at $f'(N^*)$.

$$f'(N) = r - \frac{2rN}{K} - h,$$

so

$$f'(N^*) = f'(0) = r - h.$$

This is exactly the same as the coefficient of N in our linearized equation above. This is no coincidence. Linearizing about 0 always produces the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = f'(0)N.$$

Now let's look at $N^* = K \cdot (1 - h/r)$. We want to know what happens to populations that start very close to N^*, so let's change coordinates to $x = N - N^*$ (that is, x is the distance from N to the equilibrium). This gives us the equation

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= r(x + N^*)\left(1 - \frac{x + N^*}{K}\right) - h(x + N^*) \\
&= r\left(x + N^*\right) - r\left(x + N^*\right)^2/K - h(x + N^*) \\
&= rx + rN^* - \frac{rx^2}{K} - \frac{2rN^*x}{K} - \frac{r(N^*)^2}{K} - hx - hN^* \\
&= \left(r - h - \frac{2rN^*}{K}\right)x - \frac{r}{K}x^2 + rN^* - \frac{r(N^*)^2}{K} - hN^*.
\end{aligned}$$

Notice that the constant terms are the same thing as $f(N^*)$. Since N^* is a fixed point, $f(N^*) = 0$, so we can simplify this to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(r - h - \frac{2rN^*}{K}\right)x - \frac{r}{K}x^2.$$

In addition, since we are only interested in populations very close to the equilibrium, we know that $N \approx N^*$, so $x \approx 0$. This means that x^2 is much smaller than x, so we can ignore the x^2 term. Our equation is therefore

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \left(r - h - \frac{2rN^*}{K}\right)x \\
&= \left(r - h - 2r + 2h\right)x \\
&= (h - r)x.
\end{aligned}$$

This is our standard linear equation, so we know that N^* is stable if $h - r < 0$, which means that $h < r$, and N^* is unstable if $h - r > 0$ if $h - r > 0$, so $h > r$. Just as before, we could have found this result more easily by finding $f'(x^*) = f'(0)$. Since

$$f(x) = r(x + N^*) - r(x + N^*)^2/K - h(x + N^*),$$

we know that

$$f'(x) = r - 2r(x + N^*)/K - h,$$

so

$$f'(0) = r - h - 2rN^*/K = h - r.$$

Bifurcation Diagram

A nice way to summarize these results is to plot the fixed points as a function of h. This plot is called a *bifurcation diagram*. In our case, the bifurcation diagram looks like this:

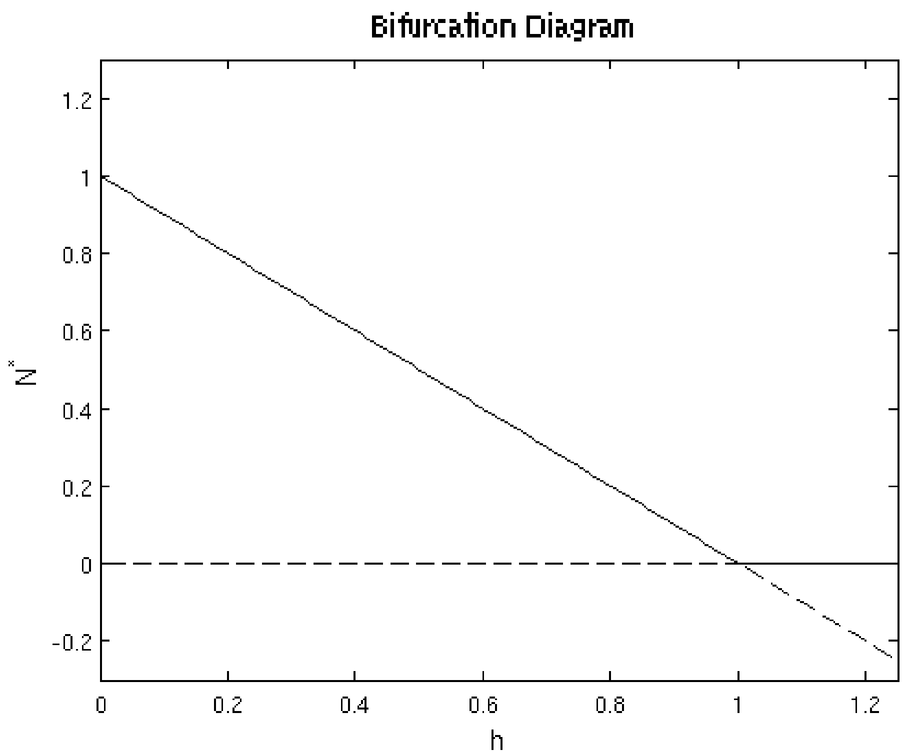

Figure 4: A bifurcation diagram for our model. Solid lines indicate a stable fixed point and dashed lines indicate an unstable fixed point. This type of bifurcation is called a transcritical bifurcation, because the two fixed points switch stability when they collide.