Homework 1

Due Friday, June 30 2017

Problem 1 (10 points)

(This is adapted from problem 5 from chapter 3 of the course textbook.)

A professor's daughter just started her freshman year at a private college that costs \$30,000 per year. When she was born (18 years ago) her grandparents put \$10,000 in a college fund for her. If the interest were compounded continuously, what annual percentage rate (APR) would the college fund need so that she has enough money today for the next four years of college tuition? What if the interest were compounded monthly? What if it were compounded yearly?

Problem 2 (10 points)

(This is adapted from problem 7 from chapter 3 of the course textbook.)

You have just signed a contract that entitles you to receive \$1,000,000 twenty years from now, but you can't wait and want your money now. Assuming that the risk-free, inflation-adjusted APR is 3% per year, compounded continuously, what is a fair price for your contract?

Problem 3 (5 points)

Let N_t be the number of rabbits in Australia at the end of year t and let M_t be the number of fox in Australia at the end of year t. Suppose that the populations interact in such a way that

$$\begin{aligned} N_{t+1} &= 6N_t - 4M_t, \\ M_{t+1} &= 2N_t. \end{aligned} \tag{1}$$

If $N_0 = M_0 = 1$, find the populations N_t and M_t for all $t \geq 0$.

Extra credit (5 points): There is a very easy way to solve the preceding problem, but it only works because we chose convenient initial conditions. A slightly more systematic way to solve the problem is by assuming N_t and M_t are of the form

$$\begin{pmatrix} N_t \\ M_t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \lambda^t \tag{2}$$

and determining the values of λ, u and v. Find a condition on N_0 and M_0 such that the solution will be of this form, and solve (1) for any such initial conditions. Show that this agrees with your solution from above.

Problem 4 (15 points)

(This is adapted from problem 4 from chapter 2 of the course textbook. Note that the text has a typo in part c – it is missing a power of 1/4.)

In a growing organism, metabolism supplies energy to both maintain existing tissues and create new tissues by cell division. Let Y_c be the metabolic rate of a single cell (that is, the amount of energy used by the cell per unit time) and let E_c be the energy required to create a new cell and let $N_c(t)$ be the total number of cells at time t. The total mass of the organism is given by

$$m(t) = m_c N_c(t), \tag{3}$$

and the total metabolic rate Y of the organism is given by

$$Y(t) = Y_c N_c(t) + E_c \frac{\mathrm{d}N_c}{\mathrm{d}t}. \tag{4}$$

It has been argued (read http://science.sciencemag.org/content/276/5309/122 or chapter 2 of our textbook for more information) that Y is related to m by the equation

$$Y(t) = Y_0 \cdot (m(t))^{3/4}, \tag{5}$$

where Y_0 is some constant.

(a) Show that equations (3), (4) and (5) can be combined to obtain

$$\frac{\mathrm{d}m}{\mathrm{d}t} = am^{3/4} - bm, \tag{6}$$

where $a \equiv Y_0 m_c / E_c$ and $b = Y_c / E_c$ are constants.

(b) Suppose that when an organism matures, its mass stops changing. That is, at maturity $m(t) \equiv M$ is constant, so $m'(t) = 0$. Find M, and show that (6) can be rewritten as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = am^{3/4}\left[1 - \left(\frac{m}{M}\right)^{1/4}\right]. \tag{7}$$

(c) Let $r = (m/M)^{1/4}$ and $R = 1 - r$. Show that (7) becomes

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\left(\frac{a}{4M^{1/4}}\right) R. \tag{8}$$

Solve this differential equation for $R(t)$ with initial condition $R(0) = R_0$ and plot $\ln\left(R(t)/R_0\right)$ vs $\tau \equiv at/(4M^{1/4})$. This plot should be a straight line with slope -1, regardless of the values of any constants.