Today, we want to extend the simple random walk to some more interesting phenomena. In particular, we will model the birth and death of individuals in an endangered species.

In particular, let $N(t)$ be the (random) # of living individuals at time t. We want our model to include random chances of birth and death.

For comparison, remember the deterministic model we looked at several weeks ago:

$$\frac{dN(t)}{dt} = bN - dN = (b-d)N,$$

where b is the birth rate and d is the death rate. This population grew exponentially w/ rate $b-d$ if $b > d$ and decayed exponentially if $b < d$. In particular, the population died out if and only if $b < d$.

As a first pass, let's use the simple random walk model we looked at earlier. If we let $P_n(t) = \Pr[N(t)=n]$, and we let β = probability of a birth in one time step and $\delta \overset{=1-\beta}{\equiv}$ probability of a death in one time step, then we have

$$P_n(t) = \frac{t!}{\left(\frac{n+t}{2}\right)!\left(\frac{t-n}{2}\right)!}\,\beta^{\frac{n+t}{2}}\,\delta^{\frac{t-n}{2}}.$$

~~To avoid problems w/ even/odd values of n and t, we will only count the population at even time steps, so we have~~

$$\cancel{P_n(t) = \frac{(2\tau)!}{(\tau+x)!\,(\tau-x)!}\,\beta^{\tau+x}\,\delta^{\tau-x}.}$$

~~(Here, $t=2\tau$ and $n=2x$.)~~

This has the obvious flaw that the population can be negative. To fix this, we will say that if $N(t)=0$ at any time t, then it stays zero afterwards.

One of the more important questions is: When will the population go extinct. To solve this, we start by assuming that the population "survives" if it ever reaches some large population N^*. Let T_n be the first time the species "survives" or goes extinct. That is,

$$T_n = \min\{t \geq 0 \text{ where } N(t) = 0 \text{ or } N(t) = N^*, \text{ given } N(0) = n\}.$$

Let P_n denote the probability of surviving if $N(0) = n$. By definition, $P_0 = 0$ and $P_{N^*} = 1$. We also have

$$P_n = \beta \cdot P_{n+1} + \delta \cdot P_{n-1}$$

This means

$$(\beta + \delta) P_n = \beta P_{n+1} + \delta P_{n-1},$$

so

$$P_n + \frac{\delta}{\beta} P_n = P_{n+1} + \frac{\delta}{\beta} P_{n-1}$$

$$\Rightarrow P_{n+1} - P_n = \frac{\delta}{\beta}\left(P_n - P_{n-1}\right).$$

In particular,

$$P_2 - P_1 = \frac{\delta}{\beta}(P_1 - P_0) = \frac{\delta}{\beta} P_1$$

$$\Rightarrow P_3 - P_2 = \frac{\delta}{\beta}\cdot(P_2 - P_1) = \frac{\delta}{\beta}\cdot\left(\frac{\delta}{\beta}P_1\right) = \left(\frac{\delta}{\beta}\right)^2 \cdot P_1$$

$$\vdots$$

$$\Rightarrow P_{n+1} - P_n = \left(\frac{\delta}{\beta}\right)^n \cdot P_1.$$

Notice that

$$P_{n+1} - P_1 = (P_{n+1} - P_n) + (P_n - P_{n-1}) + \dots + (P_2 - P_1)$$

$$= \sum_{k=1}^{n} (P_{k+1} - P_k) = \sum_{k=1}^{n} \left(\frac{\delta}{\beta}\right)^k P_1$$

$$\Rightarrow P_{n+1} = P_1 + \sum_{k=1}^{n} \left(\frac{\delta}{\beta}\right)^k P_1 = \sum_{k=0}^{n} \left(\frac{\delta}{\beta}\right)^k P_1$$

$$= \begin{cases} P_1 \cdot (n+1) & \text{if } \delta = \beta \\ P_1 \cdot \dfrac{1 - \left(\frac{\delta}{\beta}\right)^{n+1}}{1 - \left(\frac{\delta}{\beta}\right)} & \end{cases}$$

Since $P_{N^*} = 1$, we have

$$P_{N^*} = 1 = P_1 \cdot (N^*) \Rightarrow P_1 = \frac{1}{N^*} \quad \text{if} \quad \beta = \delta$$

and

$$1 = P_{N^*} = \frac{1 - (\delta/\beta)^{N^*}}{1 - \delta/\beta} \cdot P_1$$

$$\Rightarrow P_1 = \frac{1 - \delta/\beta}{1 - (\delta/\beta)^{N^*}} \quad \text{if} \quad \beta \neq \delta.$$

Therefore, If $\beta = \delta$ we have

$$P_n = \frac{n}{N^*} \quad \text{if} \quad \beta = \delta$$

and

$$P_n = \frac{1 - (\delta/\beta)^n}{1 - (\delta/\beta)^{N^*}}.$$

Therefore, the probability of 'survival' is

$$P_n = \begin{cases} \dfrac{1 - (\delta/\beta)^n}{1 - (\delta/\beta)^{N^*}} & \text{if } \beta \neq \delta \\ \dfrac{n}{N^*} & \text{if } \beta = \delta. \end{cases}$$

The probability of extinction is therefore

$$1 - P_n = \begin{cases} 1 - \dfrac{1-(\delta/\beta)^n}{1-(\delta/\beta)^{N^*}} & \beta \neq \delta \\[2ex] \dfrac{N^* - n}{N^*} & \beta = \delta \end{cases}$$

In this formula, $n = N_0$ represents the initial population and N^* is some population large enough to be safe. For an endangered species, $N_0 \ll N^*$.

Notice that if N^* is fixed, then extinction probability goes down as N_0 increases, until the population "survives" at $N_0 = N^*$ with probability one. Really, though, the population isn't safe at any finite population, so we need $N^* \to \infty$.

In this case, we have

$$P_n \to 0 \quad \text{if} \quad \delta \geq \beta \quad \text{and}$$

$$P_n \to 1 - (\delta/\beta)^n \quad \text{if } \delta < \beta,$$

So the chance of extinction is

$$\Pr[\text{extinction}] = 1 \quad \text{if} \quad \delta \geq \beta$$
$$\Pr[\text{extinction}] = \left(\delta/\beta\right)^{N_0} \quad \text{if} \quad \delta < \beta.$$

This is different from the determenistic model, where the population never went extinct if $\delta < \beta$. In contrast, the stochastic version leads to extinction with some small but positive probability even if births exceed deaths. However, this probability vanishes as $N_0 \to \infty$, so this extinction risk is only important for small populations.