Overview:

We have been looking at the population growth model described by:

$$\frac{dP_n}{dt} = \beta\cdot(n-1)\cdot P_{n-1} - (\beta+\delta)\cdot n\cdot P_n + \delta\cdot(n+1)\cdot P_{n+1}$$

$$P_n = \begin{cases} 1 & n = N_0 \\ 0 & n \neq N_0, \end{cases}$$

Where $P_n = P_r[N(t) = n]$.

We could not solve these equations directly for each P_n, so we used a generator function approach. In particular, we wrote the probability generating function

$$F(t,x) = \sum_{n=0}^{\infty} P_n(t),$$

And substituted this into our system. We obtained a (first order, linear) PDE

$$\frac{\partial F}{\partial t} + (\beta x - \delta)(1-x)\frac{\partial F}{\partial x} = 0,$$

which we solved for F.

In the end, we obtained

$$F(t,x) = \begin{cases} \left[\dfrac{\beta t + (1-\beta t)x}{1+\beta t - \beta t x} \right]^{N_0} & \text{if } \beta = \delta \\[2ex] \left[\dfrac{\delta e^{rt}\cdot(1-x) + (\beta x - \delta)}{\beta e^{rt}(1-x) + (\beta x - \delta)} \right]^{N_0} & \text{if } \beta \neq \delta \end{cases}$$

The whole purpose of finding the probability generating function was to find useful statistics about the random variable $N(t)$.

For instance, what is the average population value (or, to be more precise, the expected value of N)?

Remember from last class that

$$\langle N(t) \rangle = \left. \frac{\partial F}{\partial x} \right|_{x=1}$$

For $\beta = \delta$, we have

$$\frac{\partial F}{\partial x} = N_0 \cdot \left[\frac{\beta t + (1-\beta t)x}{1+\beta t - \beta t x} \right]^{N_0 - 1} \cdot \left[\frac{(1-\beta t)\cdot(1+\beta t - \beta t x) + \beta t \cdot (\beta t + (1-\beta t)x)}{(1+\beta t - \beta t x)^2} \right]$$

So

$$\left.\frac{\partial F}{\partial x}\right|_{x=1} = N_0 \cdot \left[\frac{\beta t}{\beta t} \right]^{N_0 - 1} \cdot \frac{(1-\beta t) + \beta t}{1^2} = N_0$$

and we have $\boxed{\langle N(t) \rangle = N_0.}$

Likewise, if $\beta \neq \delta$ we have

$$\frac{\partial F}{\partial x} = N_0 \cdot \left[\frac{\delta e^{rt} \cdot (1-x) + (\beta x - \delta)}{\beta e^{rt} \cdot (1-x) + (\beta x - \delta)} \right]^{N_0 - 1} \cdot \left[\frac{(\beta - \delta e^{rt}) \cdot (\beta e^{rt} \cdot (1-x) + \beta x - \delta) - (\beta - \beta e^{rt}) \cdot (\delta e^{rt}(1-x) + \beta x - \delta)}{(\beta e^{rt} \cdot (1-x) + (\beta x - \delta))^2} \right]$$

and

$$\frac{\partial F}{\partial x} = N_0 \cdot \left[\frac{\beta - \delta}{\beta - \delta} \right]^{N_0 - 1} \cdot \frac{(\beta - \delta e^{rt}) \cdot (\beta - \delta) - (\beta - \beta e^{rt}) \cdot (\beta - \delta)}{(\beta - \delta)^2}$$

$$= N_0 \cdot \frac{(\beta-\delta)\left[\beta-\delta e^{rt}-\beta+\beta e^{rt}\right]}{(\beta-\delta)^2}$$

$$= N_0 \cdot \frac{\beta e^{rt}-\delta e^{rt}}{(\beta-\delta)}$$

$$= N_0 e^{rt}.$$

Therefore,

$$\boxed{\langle N(t)\rangle = N_0 e^{rt},} \quad \text{where } r=\beta-\delta.$$

This formula actually also works when $\beta=\delta$.

This agrees perfectly with our original deterministic model for exponential growth. That is, $\langle N(t)\rangle$ satisfies

$$\boxed{\frac{d\langle N(t)\rangle}{dt} = (\beta-\delta)\langle N(t)\rangle, \quad \langle N(0)\rangle = N_0}$$

Of course, we expect some differences between the two models. This is the average behavior, but the stochastic model doesn't have to follow the average - there will be some spread.

This spread is measured by the variance (or standard deviation). Remember from last class that

$$\operatorname{Var}[N(t)] = \left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial x} - \left(\frac{\partial F}{\partial x}\right)^2 \right]_{x=1},$$

and

$$\sigma(N(t)) = \sqrt{\operatorname{Var}[N(t)]}.$$

The computation of $\frac{\partial^2 F}{\partial x^2}$ is somewhat long (but straightforward), so we will just give the result:

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_{x=1} = \begin{cases} 2\beta N_0 t + N_0^2 - N_0, & \text{if } \beta = \delta \\ (\beta+\delta) N_0 \cdot \left(\dfrac{e^{rt}-1}{r}\right) \cdot e^{rt} + N_0^2 e^{2rt} - N_0 e^{rt} \end{cases}$$

This means that

$$\operatorname{Var}[N(t)] = \begin{cases} 2\beta N_0 t & \text{if } \beta = \delta \\ (\beta+\delta) N_0 \cdot \left(\dfrac{e^{rt}+1}{r}\right) \cdot e^{rt} & \text{if } \beta \neq \delta \end{cases}$$

This means that

$$\sigma[N(t)] = \begin{cases} \sqrt{2\beta N_0 t} & \text{if } \beta = \delta \\ \sqrt{(\beta+\delta) N_0 e^{rt}\left(\frac{e^{rt}-1}{r}\right)} & \text{if } \beta \neq \delta \end{cases}$$

Notice that, for fixed N_0, β and δ,

$\sigma[N(t)] \propto \sqrt{t}$ if $\beta = \delta$, so the standard deviation grows without bound, but fairly slowly.

On the other hand, if $\beta \neq \delta$ then for large t we have

$$\sigma[N(t)] \propto e^{rt}.$$

(This is usually written as $\sigma[N(t)] = O(e^{rt})$.)

For $\beta > \delta$, the standard deviation grows without bound and goes exponentially fast. For $\beta < \delta$, the standard deviation shrinks to zero.

Remember, the standard deviation is not an ideal measure of the quality of our average estimate. For instance, when $\beta > \delta$ the standard deviation goes to infinity, but so does the mean. This suggests that our relative error might not be too bad. Likewise, when $\beta < \delta$ the s.d. goes to zero, but so does the mean. The relative error in this case might be fairly bad. To check, we should calculate the coefficient of variation:

$$C_v[N(t)] = \frac{\sigma[N(t)]}{\langle N(t) \rangle}$$

$$C_v = \begin{cases} \sqrt{\dfrac{2\beta t}{N_0}} & \text{if } \beta = \delta \\[2ex] \sqrt{\dfrac{(\beta+\delta)(e^{rt}-1)}{N_0 r e^{rt}}} = \sqrt{\dfrac{(\beta+\delta)(1-e^{-rt})}{N_0 r}} & \text{if } \beta \neq \delta \end{cases}$$

This gives some interesting conclusions! If $\beta > \delta$, so $r > 0$, we expect the population to grow without bound. Our absolute error becomes infinite, which is obviously undesirable, but our relative error approaches a constant.

In particular, if $\beta > \delta$ then

$$\lim_{t\to\infty} C_V[N(t)] = \sqrt{\frac{\beta+\delta}{N_0(\beta-\delta)}}.$$

If N_0 is large, then this constant is small. This means that if the initial population is sufficiently large, we have a very good (by relative error) estimate of $N(t)$.

Oddly, if $\beta < \delta$, so $r < 0$, the C_V goes to infinity. This means that our absolute error goes to zero (we can be very sure the population is going extinct) but the relative error actually blows up.

In the special case where $\beta = \delta$, both measures of error go to infinity.

Finally (and, at least for small populations) we want to know the probability of being extinct.

Extinction means that the population is zero so $P_0(t)$ is the probability that the population has gone extinct by time t.

Remember that $P_0(t) = F(t, 0)$, so

$$P_0(t) = \begin{cases} \left[\dfrac{\beta t}{1+\beta t}\right]^{N_0} & \text{if } \beta = \delta \\ \left[\dfrac{\delta e^{rt} - \delta}{\beta e^{rt} - \delta}\right]^{N_0} & \text{if } \beta \neq \delta. \end{cases}$$

In particular, the probability that the population eventually goes extinct is given by

$\lim_{t\to\infty} P_0(t)$. If $\beta = \delta$, then

$$\lim_{t\to\infty} P_0(t) = \lim_{t\to\infty} \left[\frac{\beta t}{1+\beta t}\right]^{N_0} = 1^{N_0} = 1.$$