Overview:

We have been looking at a simple model for a pendulum given by

$$\ddot{\theta} = -a\dot{\theta} - b\sin(\theta)$$

where $\theta$ is the angle between the pendulum rod and vertical, $a$ is a measure of friction and $b$ is a measure of gravity. We used the change of variables $x_1 = \theta$ and $x_2 = \dot{\theta}$ to rewrite this as the first order system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -ax_2 - b\sin(x_1). \end{cases}$$

This system has equilibria (places where neither $x_1$ nor $x_2$ are changing) at

$$x_1^* = n\pi, \quad x_2^* = 0$$

We found that if $x_1$ and $x_2$ were sufficiently close to $x_1^* = x_2^* = 0$, then our system was approximately
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_2 - bx_1. \end{cases}$$

Which had either a stable spiral or node at the origin, depending on the size of $a$.

If we need to linearize about each equilibrium individually, we will be here a long time. In order to be more systematic, we will divide the equilibria into two cases: $x_1^* = 2K\pi$ and $x_1^* = (2K+1)\pi$, where $K$ is an integer. These correspond to the pendulum hanging straight down or standing straight up.

First, choose an integer $K$ and consider solutions close to $x_1^* = 2K\pi$, $x_2^* = 0$. That is, we want $|x_1(t) - 2K\pi| \ll 1$ and $|x_2(t)| \ll 1$. This suggests we change variables to $y_1 = x_1 - 2K\pi$ and $y_2 = x_2$. We then have $\dot{x}_1 = \dot{y}_1$ and $\dot{x}_2 = \dot{y}_2$, so

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -ay_2 - b\sin(y_1 + 2K\pi). \end{cases}$$

We have $|y_1|, |y_2| \ll 1$, so we can taylor expand about $y_1 = y_2 = 0$. The $y_2$ terms are already linear, so we really only need to taylor expand $\sin(y_1 + 2K\pi)$.

If we let $f(x) = \sin(x + 2k\pi)$, we can write

$$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + O(x^4)$$

Since
$$f(0) = \sin(2k\pi) = 0,$$
$$f'(0) = \cos(2k\pi) = 1,$$
$$f''(0) = -\sin(2k\pi) = 0,$$
$$f^{(3)}(0) = -\cos(2k\pi) = -1,$$

we have

$$f(x) \quad x - \frac{x^3}{6} + O(x^4)$$

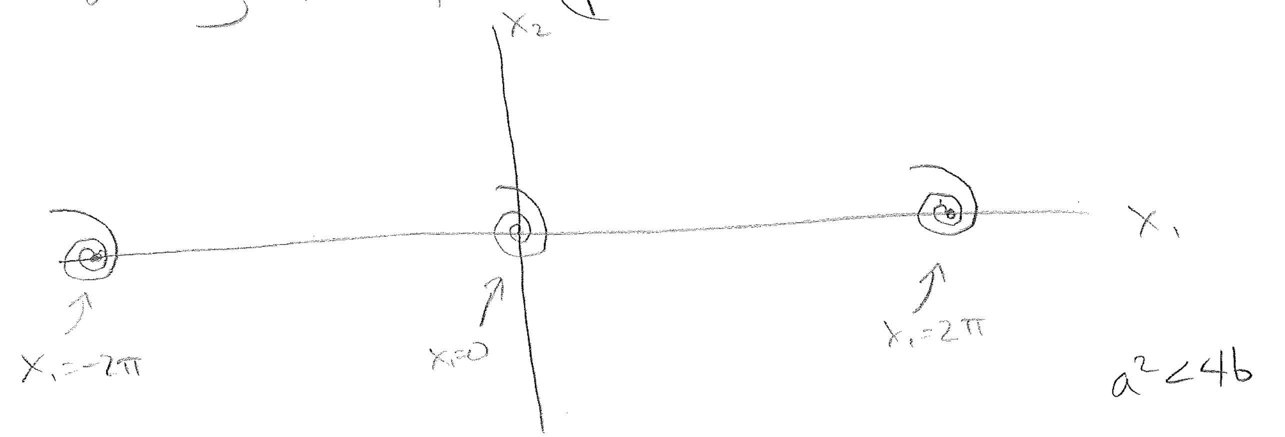
so

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -ay_2 - by_1 + \frac{b}{6}y_1^3 + O(y_1^4). \end{cases}$$

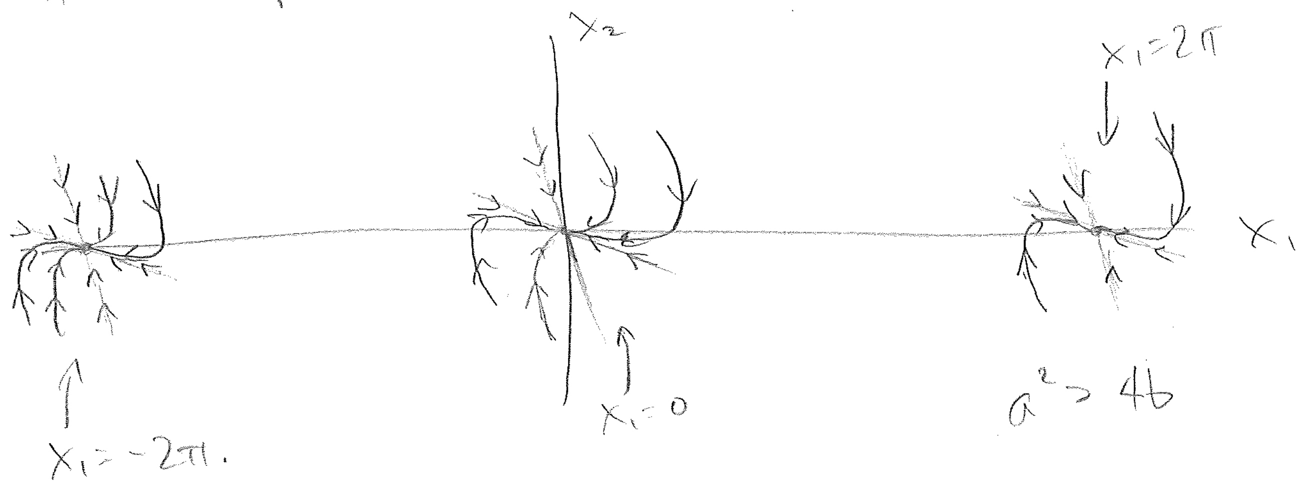
Since $y_1$ is very small, higher powers of $y_1$ are very very small, so this is approximately

$$\begin{matrix} \dot{y}_1 = y_2 \\ \dot{y}_2 = -ay_2 - by_1 \end{matrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \vec{y}.$$

We already know how to solve this linear system. If $a^2 < 4b$, then the system has a stable spiral at $y_1 = y_2 = 0$. Remember, $y_1$ and $y_2$ are just shifted coordinates for $x_1$ and $x_2$, so $x_1$ and $x_2$ must also have a stable spiral at $x_1 = 2k\pi$, $x_2 = 0$. We get the (partial) phase portrait




Similarly, if $a^2 > 4b$, the linearized system has a stable node at $y_1 = y_2 = 0$, so the full system has a stable node at $x_1 = 2k\pi$ and $x_2 = 0$. We get the (partial) phase portrait



The other equilibria are at $x_1^* = (2K+1)\pi$, $x_2^* = 0$. Since we are interested in solutions that are very close to these equilibria, we want $|x_1 - (2K+1)\pi| << 1$ and $|x_2| << 1$. With this in mind, we will let

$y_1 = x_1 - (2K+1)\pi$ and $y_2 = x_2$. We therefore have $\dot{y}_1 = \dot{x}_1$ and $\dot{y}_2 = \dot{x}_2$, so

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -ay_2 - b\sin(y_1 + (2K+1)\pi). \end{cases}$$

As before, we need to Taylor expand about $y_1 = 0$.

If $f(x) = \sin(x + (2K+1)\pi)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + O(x^4).$$

We have

$$f(0) = \sin((2K+1)\pi) = 0,$$
$$f'(0) = \cos((2K+1)\pi) = -1,$$
$$f''(0) = -\sin((2K+1)\pi) = 0,$$
$$f^{(3)}(0) = -\cos((2K+1)\pi) = 1,$$

so

$$\sin(x + (2K+1)\pi) = -x + \frac{x^3}{6} + O(x^4).$$

We therefore have

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -ay_2 + by_1 - \frac{b}{6}y_1^3 + O(y_1^4).$$

since $y_1$ is very small, the higher order terms are close to zero, so we have approximately

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -ay_2 + by_1.$$

This is the linear system

$$\dot{\vec{y}} = \begin{pmatrix} 0 & 1 \\ b & -a \end{pmatrix} \vec{y}.$$

To see what these solutions look like, we need the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 \\ b & -a \end{pmatrix}$.
The eigenvalues can be found by setting
$\det(A - \lambda I) = 0$, so

$$\det\begin{pmatrix} -\lambda & 1 \\ b & -a-\lambda \end{pmatrix} = -\lambda(-a-\lambda) - b = \lambda^2 + a\lambda - b = 0.$$

Therefore,

$$\lambda = \frac{-a \pm \sqrt{a^2+4b}}{2}$$

Since $a$ and $b$ are both positive $a^2+4b>0$, so both $\lambda$'s are always real. Moreover, $a^2+4b>a^2$, so $\sqrt{a^2+4b}>a$. This means that $-a+\sqrt{a^2+4b}>0$, but $-a-\sqrt{a^2+4b}<0$, so we must have a saddle.

To find the eigenvectors, we need to solve

$$\begin{pmatrix} -\lambda & 1 \\ b & -a-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with each $\lambda$ plugged in. Fortunately, this system must be degenerate, so we can just solve one of the equations. For instance,

$$-\lambda v_1 + v_2 = 0 \Rightarrow v_2 = \lambda v_1$$

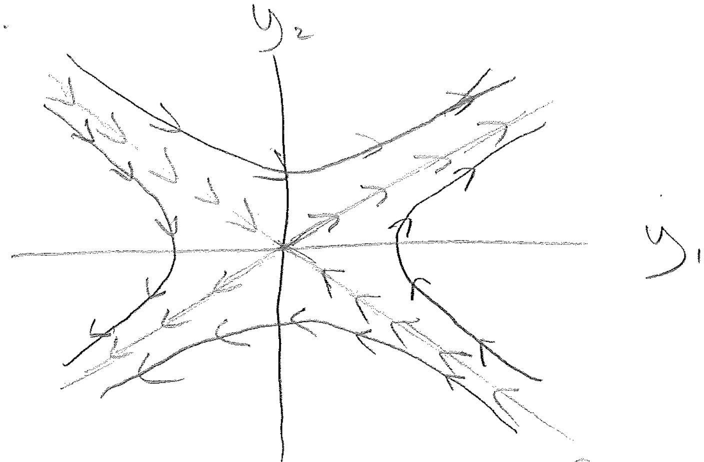
$$\Rightarrow \vec{v} = c\begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

This means that

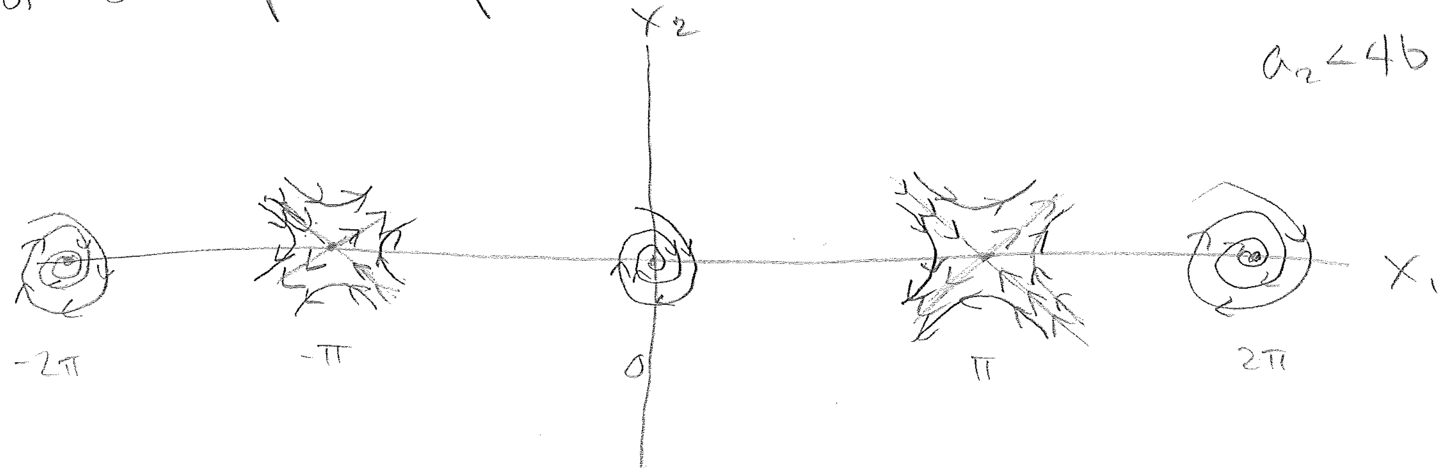
$$\lambda_1 = \frac{-a+\sqrt{a^2+4b}}{2} > 0 \quad \text{and} \quad \vec{v}_1 = c_1\begin{pmatrix} 1 \\ \frac{-a+\sqrt{a^2+4b}}{2} \end{pmatrix} \quad \text{and}$$

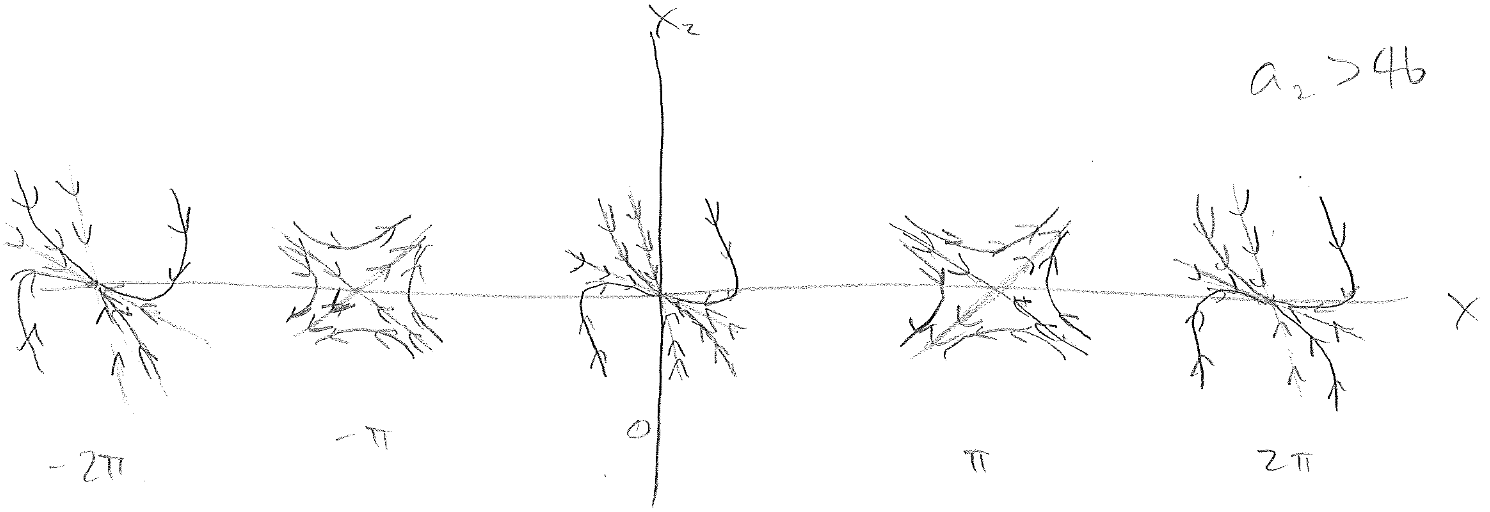
$$\lambda_2 = \frac{-a-\sqrt{a^2+4b}}{2} < 0 \quad \text{and} \quad \vec{v}_2 = c_2\begin{pmatrix} 1 \\ \frac{-a-\sqrt{a^2+4b}}{2} \end{pmatrix}.$$

The phase plane for this linear system is therefore:




The full system must therefore have saddles at each $x_1^* = (2k+1)\pi$, $x_2^* = 0$. We can fill in more of our phase portraits:

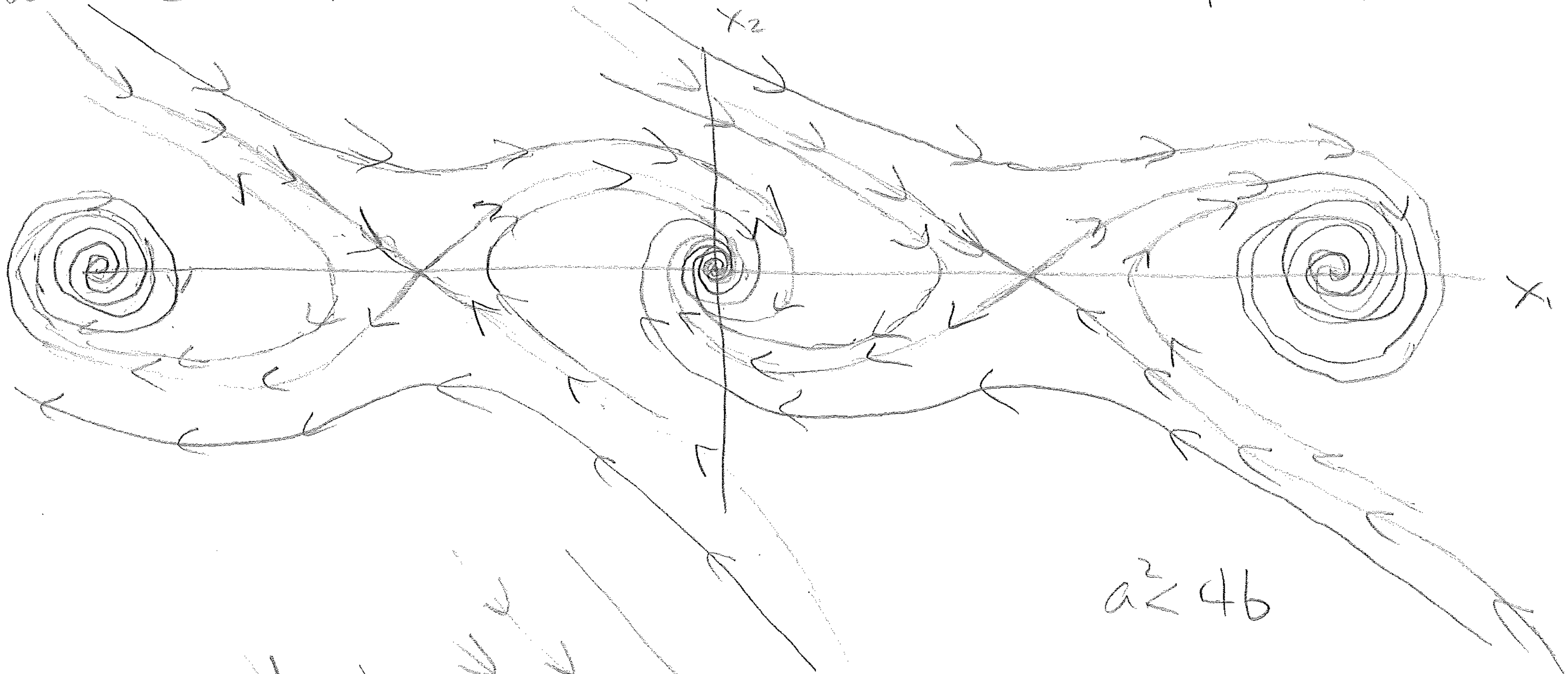


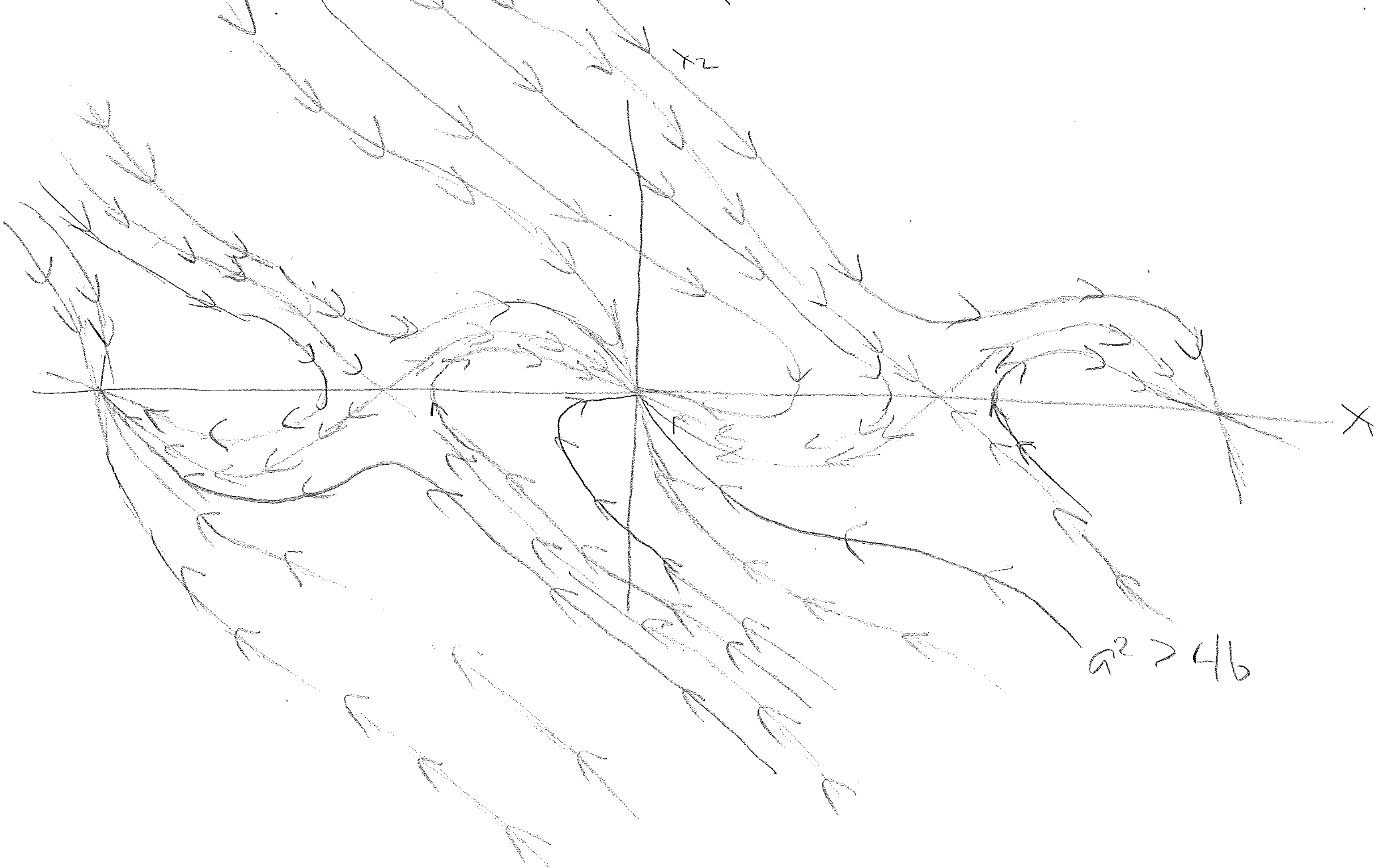
This partial phase portrait only tells us what
happens very close to equilibria, but that is
already quite a bit of information. For instance,
we know that if the pendulum is almost straight down
and not moving too fast, it will move towards straight
down with/without oscillations (depending on a) and
not make a full revolution. If it starts very
close to straight up and not moving too fast, it
will almost always start to fall back down. If
the initial push is too small, the pendulum won't make it to
vertical and will fall back the same way it came.
If the push is too big, the pendulum will pass
through vertical and then fall the other way. There
is also a perfect amount of push to get
the pendulum to vertical and make it stay.

We want more, though. We want to know
what happens everywhere in phase space, not just
near equilibria. Fortunately, we know that
nearby solutions must behave similarly and two solutions
can never cross.

With those facts (and some appeals to symmetry), we can fill in the rest of the phase plane.




$a^2 < 4b$




$a^2 > 4b$