Calculus overview:

Remember, we are looking at the differential equation

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2), \end{cases}$$

where f_1 and f_2 are nonlinear functions. We want to linearize f_1 and f_2 near equilibria (points where both $\dot{x}_1$ and $\dot{x}_2$ are zero).

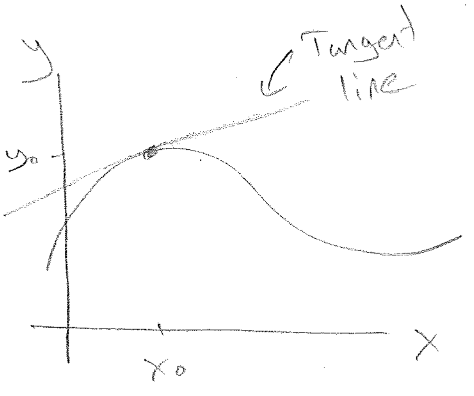
As a reminder, suppose we have a function of one variable $y = f(x)$ and a point x_0 such that $f(x_0) = y_0$. The "best linear approximation" of f near x_0 is the tangent line to $f(x)$ that passes through (x_0, y_0). This equation is given by

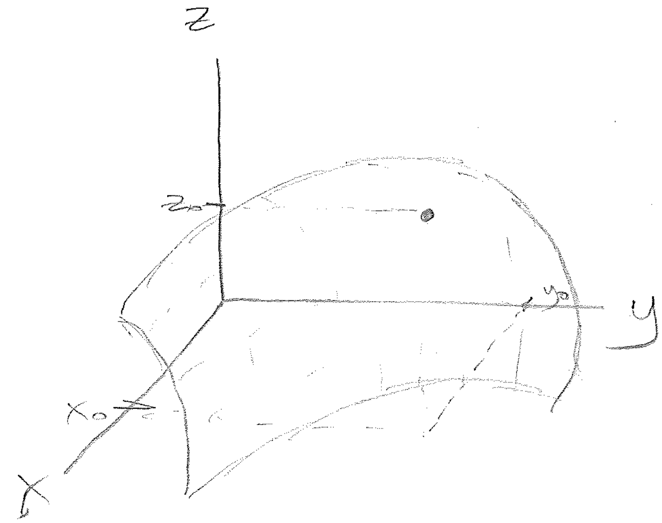
$$y - y_0 = \left.\frac{df}{dx}\right|_{x_0} \cdot (x - x_0), \quad \text{or}$$

$$\left.\frac{df}{dx}\right|_{x_0} \cdot (x - x_0) - (y - y_0) = 0.$$

Now suppose we have a function of two variables $z = f(x, y)$ and a point x_0, y_0 with $z_0 = f(x_0, y_0)$.

The "best linear approximation" of f near (x_0, y_0) is the tangent plane to f(x) that passes through (x_0, y_0, z_0).





Remember, the equation for a plane passing through (x_0, y_0, z_0) is

$$a \cdot (x - x_0) + b \cdot (y - y_0) + c \cdot (z - z_0) = 0,$$

where the vector $\langle a, b, c \rangle$ is perpendicular to the plane. This means that we need a vector perpendicular to the tangent plane. To find this, note that every curve we can draw on the surface $z = f(x, y)$ that passes through (x_0, y_0, z_0) is also tangent to the plane.

With that in mind, let $(x(t), y(t), z(t))$ be a curve on the surface that passes through (x_0, y_0, z_0). The fact that the curve is on our surface means

$$z(t) = f(x(t), y(t)), \qquad (1)$$

and the fact that the curve passes through (x_0, y_0, z_0) means there is a time t_0 such that

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0.$$

If we take the derivative of (1) with respect to t (using the chain rule) we get

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}, \quad \text{so}$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} - \frac{dz}{dt} = 0 \quad \text{and}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \left.\frac{dx}{dt}\right|_{t_0} + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \left.\frac{dy}{dt}\right|_{t_0} - \left.\frac{dz}{dt}\right|_{t_0} = 0.$$

This means that

$$\left\langle \frac{\partial f}{\partial x}\Big|_{x_0,y_0}, \frac{\partial f}{\partial y}\Big|_{x_0,y_0}, -1 \right\rangle \cdot \left\langle x'(t_0), y'(t_0), z'(t_0) \right\rangle = 0,$$

so these two vectors are perpendicular. The vector $\langle x'(t_0), y'(t_0), z'(t_0)\rangle$ is the tangent vector to the curve, so our tangent plane must be parallel to it. This means that

$$\left\langle \frac{\partial f}{\partial x}\Big|_{x_0,y_0}, \frac{\partial f}{\partial y}\Big|_{x_0,y_0}, -1 \right\rangle$$

is perpendicular to our tangent plane. Therefore, the tangent plane is

$$\frac{\partial f}{\partial x}\Big|_{x_0,y_0}\cdot(x-x_0) + \frac{\partial f}{\partial y}\Big|_{x_0,y_0}\cdot(y-y_0) - (z-z_0) = 0,$$

or

$$\boxed{z = z_0 + \frac{\partial f}{\partial x}\Big|_{x_0,y_0}\cdot(x-x_0) + \frac{\partial f}{\partial y}\Big|_{x_0,y_0}\cdot(y-y_0)}$$

Now let's go back to our system of ODEs.

We have

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

and $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$.

The "best linear approximation" to this system is given by replacing f_1 and f_2 with their tangent planes. Using our previous result, we get

$$\begin{cases} \dot{x}_1 = f_1(x_1^*, x_2^*) + \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1^*, x_2^*} \cdot (x_1 - x_1^*) + \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1^*, x_2^*} \cdot (x_2 - x_2^*) \\ \dot{x}_2 = f_2(x_1^*, x_2^*) + \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1^*, x_2^*} \cdot (x_1 - x_1^*) + \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1^*, x_2^*} \cdot (x_2 - x_2^*). \end{cases}$$

We know $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$, so the constant terms both drop out. If we let $y_1 = x_1 - x_1^*$ and $y_2 = x_2 - x_2^*$, then we have

$$\begin{cases} \dot{y}_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1^*, x_2^*} \cdot y_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1^*, x_2^*} \cdot y_2 \\ \dot{y}_2 = \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1^*, x_2^*} \cdot y_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1^*, x_2^*} \cdot y_2 \end{cases}$$

We will write

$$Df(x_1^*, x_2^*) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1^*, x_2^*} & \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1^*, x_2^*} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1^*, x_2^*} & \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1^*, x_2^*} \end{pmatrix},$$

which is usually called the Jacobian matrix (or just the Jacobian) of f.

Our linearized system is therefore

$$\dot{\vec{y}} = Df(x_1^*, x_2^*) \cdot \vec{y}.$$

The equilibrium (x_1^*, x_2^*) is therefore stable if all the eigenvalues of $Df(x_1^*, x_2^*)$ have negative real part and it is unstable if any eigenvalue of $Df(x_1^*, x_2^*)$ has positive real part.

As a test, remember our spring model

$$\begin{cases} \dot{x}_1 = x_2 & = f_1(x_1, x_2) \\ \dot{x}_2 = -a x_2 - b\sin(x_1) & = f_2(x_1, x_2) \end{cases}$$

with equilibria $x_1^* = n\pi$, $x_2^* = 0$.

We have

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_2}{\partial x_1} = -b\cos(x_1) \text{ and}$$

$$\frac{\partial f_2}{\partial x_2} = -a.$$

The Jacobian of f is therefore

$$Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -b\cos(n\pi) & -a \end{pmatrix}.$$

If n is even, this is

$$Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

and if n is odd, this is

$$Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ b & -a \end{pmatrix},$$

Which matches our old results.

These results also work equally well in more than two dimensions. We won't go higher than three in class, so for a system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) \end{cases}$$

with equilibrium (x_1^*, x_2^*, x_3^*), the Jacobian matrix is

$$Df(x_1^*, x_2^*, x_3^*) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{x^*} & \left.\frac{\partial f_1}{\partial x_2}\right|_{x^*} & \left.\frac{\partial f_1}{\partial x_3}\right|_{x^*} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{x^*} & \left.\frac{\partial f_2}{\partial x_2}\right|_{x^*} & \left.\frac{\partial f_2}{\partial x_3}\right|_{x^*} \\ \left.\frac{\partial f_3}{\partial x_1}\right|_{x^*} & \left.\frac{\partial f_3}{\partial x_2}\right|_{x^*} & \left.\frac{\partial f_3}{\partial x_3}\right|_{x^*} \end{pmatrix}.$$

The stability rules are exactly the same.

Example:

Now let's look at a more interesting problem. Suppose we have a chemical reaction where a molecule S (called the substrate) is converted to a new molecule P (called the product) by means of an enzyme E. That is, the substrate first attaches itself to the enzyme, then changes into the product P and detaches. This process is usually written as

$$E + S \rightarrow ES \rightarrow E + P.$$

In principle, every chemical reaction is reversible, so we should allow both reactions to go backwards. We will assume that the product is removed as soon as it is produced, so it never turns back into ES or S, but we will allow the substrate to detach from the enzyme without changing into P. That is we will write

$$E + S \rightleftharpoons ES \rightarrow E + P.$$

We will also assume that the substrate is being produced (or pumped in) at a constant rate. We would like to know how fast the product P is produced.

To model this process we will make a few simplifying assumptions. First, we will assume that our tank is well-mixed, so that we don't have to worry about the locations of individual molecules. Instead, we can just measure concentrations. With that in mind, let $x=[S]$ be the concentration of substrate, $y=[E]$ be the concentration of enzyme and $z=[ES]$ be the concentration of enzyme-substrate complex.