Recap:

Formula for Mortgage payments

$$P(t+\Delta t) = (1+r\Delta t)P(t) - M, \text{ with } P(0) = P_0$$

We first said that r, Δt and M are constant. This works just fine, and we obtained

$$P(t) = (1+r\Delta t)^{t/\Delta t} P_0 + \frac{1-(1+r\Delta t)^{t/\Delta t}}{r\Delta t} M.$$

How easy is this to interpret? Here are some things that should be true (are they obvious?)

- If the interest rate r goes up, so should P.
- If the payment M goes up, P should go down.

From this equation, can you tell what would happen if Δt got smaller/larger?
How long will it take for the loan to be paid off?
Can you always pay off your loan (for any choice of parameters)?

This equation's not too bad; we can answer all these questions, but they're not entirely straightforward. Many problems are much more complicated than this.

Potential Solution:

Look for a parameter that's very small and let it go to zero (or look for one that's very big and let it go to ∞). At the very least, this removes one parameter from the model, and often it makes analysis much easier.

In this case, we can't guarantee that M or P_0 are small, and we hope that r is not small, but Δt might be.

Small compared to what?

$\Delta t \approx 1$ month. How do we decide if that's small?

The question doesn't really make sense. It is only small or large when compared to something. In our case, it seems reasonable to compare Δt to T, the lifetime of the loan. In general, quantities with units aren't small or large in an absolute sense, but the ratio of two quantities w/ the same units can meaningfully be small/large.

In our case, $\dfrac{\Delta t}{T} = \dfrac{1/12 \text{ yrs}}{30 \text{ yrs}} \approx .0028 \ll 1$.

Whether this is small enough is, ultimately, a judgement call on the part of the modeler.

1st attempt:

$$P(t+\Delta t) = (1+r\,\Delta t)P(t) - M$$

We want to take the limit as Δt goes to zero.
To deal with $P(t+\Delta t)$ and $P(t)$, we need to get some sort of derivative term, so we'll simplify like so:

$$P(t+\Delta t) = P(t) + r\,\Delta t\,P(t) - M$$

$$\Rightarrow \quad P(t+\Delta t) - P(t) = r\,\Delta t\,P(t) - M$$

$$\Rightarrow \quad \frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t) - \frac{M}{\Delta t}$$

The $\lim_{\Delta t \to 0}$ of the left side is $P'(t)$, but the $\lim_{\Delta t \to 0}$ of the right side is infinite — it doesn't exist.

2nd attempt:

There is a problem with M when we take this limit. Intuitively, we can see the problem right away: If you pay off your loan twice as often, you should only have to pay $\approx$ half as much each time. That is, M shouldn't be constant w/ respect to Δt. In particular, we expect $M \approx C \cdot \Delta t$ for some constant C.

Which constant? Well, if we only make one payment, then $\Delta t = T$, and the total payment must be $\approx P_0$, so $P_0 = CT \Rightarrow C = \frac{P_0}{T}$. This means

$$M = \frac{P_0}{T} \Delta t.$$

Therefore,

$$P(t+\Delta t) = (1+r\Delta t)P(t) - \frac{P_0}{T}\Delta t$$

$$\Rightarrow \frac{P(t+\Delta t) - P(t)}{\Delta t} = rP(t) - \frac{P_0}{T}$$

This means

$$\lim_{\Delta t \to 0}\left[\frac{P(t+\Delta t) - P(t)}{\Delta t}\right] = \lim_{\Delta t \to 0}\left[rP(t) - \frac{P_0}{T}\right]$$

$$\Rightarrow \frac{dP}{dt} = rP - \frac{P_0}{T}.$$

2nd attempt (cont.):

$$\frac{dP}{dt} = rP - \frac{P_0}{T} \Rightarrow \frac{dP}{dt} - rP = -\frac{P_0}{T}$$

Many ways to solve this: variation of parameters, integrating factor, separable, homogeneous/particular solutions, etc.

Let's use an integrating factor. Remember, if

$$y'(t) + p(t)y(t) = q(t),$$

then we can multiply both sides by $e^{\int p(t)\,dt}$ and get

$$\left[e^{\int p(t)\,dt} y'(t) + p(t)e^{\int p(t)} y(t)\right] = e^{\int p(t)\,dt} q(t)$$

$$\Rightarrow \frac{d}{dt}\left[e^{\int p(t)\,dt} y(t)\right] = e^{\int p(t)\,dt} q(t).$$

In our case, $p(t) = -r$ and $q(t) = -\frac{P_0}{T}$ (and $y = P$), so

$$\int \frac{d}{dt}\left[e^{-rt} P(t)\right] = \int e^{-rt} \cdot -\frac{P_0}{T}$$

$$\Rightarrow e^{-rt} P(t) = \frac{P_0}{rT} e^{-rt} + c$$

$$\Rightarrow P(t) = \frac{P_0}{rT} + ce^{rt}.$$

Since $P(0) = P_0$,

$$P_0 = \frac{P_0}{rT} + c \Rightarrow c = P_0 - \frac{P_0}{rT} = \frac{P_0(rT-1)}{rT}$$

$$\Rightarrow \boxed{P(t) = \frac{P_0}{rT} + \frac{P_0(rT-1)}{rT} e^{rt}}$$

Remember P_0, r and T are constants.

Can we answer our original questions more easily?

If r goes up, then the $\frac{1}{rT}$ term brings down P, but the e^{rt} brings up P. For reasonably large t, the exponential is more important, so $P(t)$ goes up.

Remember that $M = \frac{P_0 \Delta t}{T}$, so M goes up when P_0 increases or T decreases. Both of these increase P.

As a final test, how long does it take for the loan to be paid off? it should be $\approx T$, but not exactly.

$$P(t) = 0 \Rightarrow \frac{P_0}{rT} + P_0\frac{(rT-1)}{rT} e^{rt} = 0$$

$$\Rightarrow 1 + (rT-1)e^{rt} = 0$$

$$\Rightarrow 1 = (1-rT)e^{rt}$$

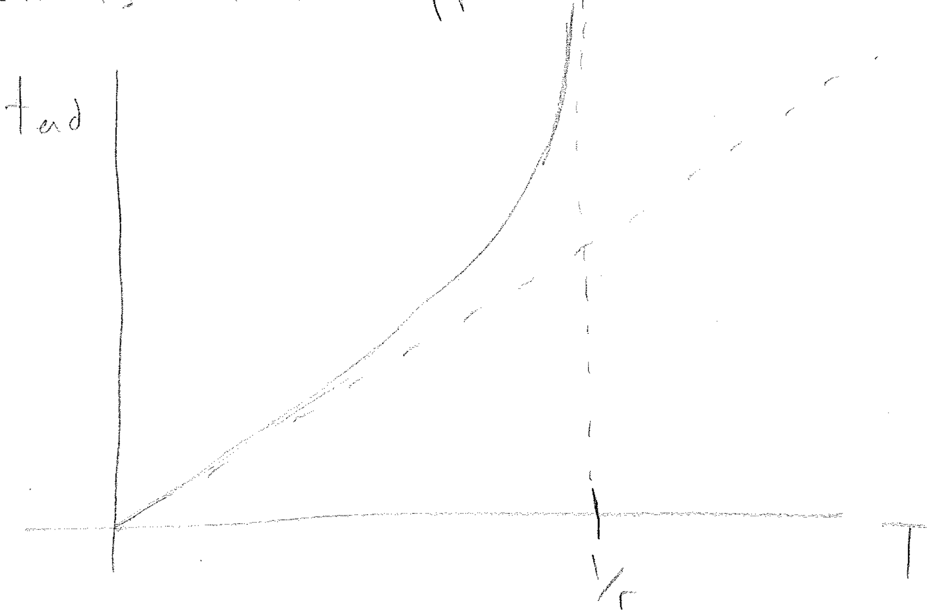
$$\Rightarrow \frac{1}{1-rT} = e^{rt}$$

$$\Rightarrow \ln\left(\frac{1}{1-rT}\right) = \cancel{e}^{rt}$$

$$\Rightarrow \frac{1}{r}\ln\left(\frac{1}{1-rT}\right) = t$$

$$\Rightarrow t_{end} = -\frac{1}{r}\ln(1-rT)$$

How is this approximation



This is pretty good when T is a lot smaller than $1/r$, pretty bad when T gets too close to $1/r$, and doesn't even work when $T > 1/r$. That's because if $T > 1/r$, then $rT - 1 > 0$, so $P(t)$ is always increasing. that means you can't keep up with the interest by making payments of $M = \frac{P_0}{T}\Delta t$.

3rd attempt:

Here's a better way to think about things:

We said it didn't make sense to talk about Δt being small, just $\frac{\Delta t}{T}$, where T is a characteristic time-scale like time-to-pay-off-loan. The reason our first limit didn't work is that we made M constant, but if M is constant then T depends on Δt. In fact, we found that

$$M = P_0 r \Delta t \left(\frac{(1+r\Delta t)^{T/\Delta t}}{(1+r\Delta t)^{T/\Delta t} - 1} \right).$$

In particular, if $\Delta t \to 0$ but we want M constant, then $T \to 0$ as well, so $\frac{\Delta t}{T}$ doesn't get small.

Approximating $M \approx \frac{P_0}{T} \Delta t$ is the same as saying

$$\frac{1}{T} \approx r \left(\frac{(1+r\Delta t)^{T/\Delta t}}{(1+r\Delta t)^{T/\Delta t} - 1} \right).$$

It turns out that this approximation isn't bad when Δt and rT are small, but gets much worse when rT is large.

3rd attempt:

What if we don't approximate at all?

We have

$$P(t+\Delta t) = (1+r\Delta t)P(t) - M$$

$$\Rightarrow P(t+\Delta t) = (1+r\Delta t)P(t) - P_0 r \Delta t \left(\frac{(1+r\Delta t)^{T/\Delta t}}{(1+r\Delta t)^{T/\Delta t} - 1} \right)$$

$$\Rightarrow \frac{P(t+\Delta t) - P(t)}{\Delta t} = rP(t) - P_0 r \left(\frac{(1+r\Delta t)^{T/\Delta t}}{(1+r\Delta t)^{T/\Delta t} - 1} \right)$$

$$\Rightarrow \frac{dP}{dt} = rP(t) - \frac{rP_0 e^{rT}}{e^{rT} - 1}.$$

This is the same form as before, but now

$$q(t) = \frac{-rP_0 e^{rT}}{e^{rT}-1},$$

so $$\frac{d}{dt}\left[e^{-rt}P(t)\right] = e^{-rt} \cdot \frac{-rP_0 e^{rT}}{e^{rT}-1}$$

$$\Rightarrow e^{-rt}P = e^{-rt} \cdot \frac{P_0 e^{rT}}{e^{rT}-1} + C$$

$$\Rightarrow P = \frac{P_0 e^{rT}}{e^{rT}-1} + Ce^{rt}$$

3rd attempt (cont.):

Since $P(0) = P_0$

$$P_0 = \frac{P_0 e^{rT}}{e^{rT}-1} + C$$

$$\Rightarrow \frac{P_0(e^{rT}-1)}{e^{rT}-1} - \frac{P_0 e^{rT}}{e^{rT}-1} = C$$

$$\Rightarrow \frac{-P_0}{e^{rT}-1} = C$$

$$\Rightarrow P(t) = \frac{P_0 e^{rT}}{e^{rT}-1} - \frac{P_0}{e^{rT}-1} e^{rt}$$

$$\Rightarrow P(t) = \frac{P_0}{e^{rT}-1}\left[e^{rT} - e^{rt}\right],$$

(Notice that now $P(T) = 0$, as expected.)