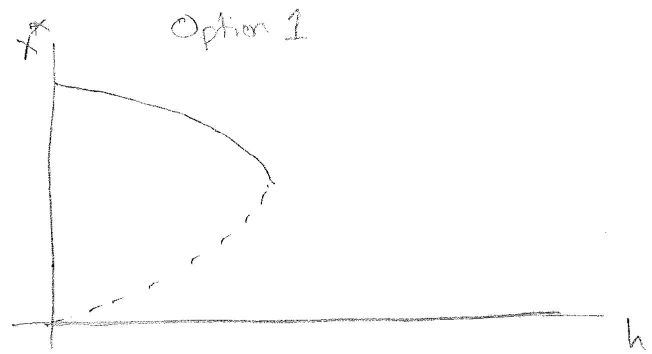
Overview:

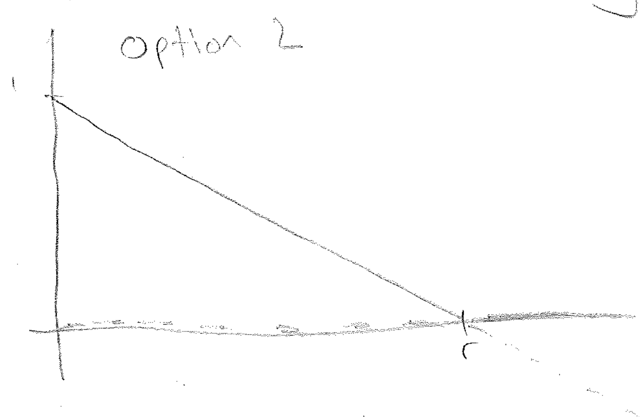
Remember our last two harvest models:

1) $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H \Rightarrow \dot{x} = rx(1-x) - h$

2) $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qTN \Rightarrow \dot{x} = rx(1-x) - hx$

We can summarize our analysis in two bifurcation diagrams:





We said that option 1 was more "sensitive to perturbations of h" because a small change in h could lead to a drastic change in x^*. This notion of sensitivity describes what happens if we keep the same equation but modify the parameters.

There's another notion of "sensitivity" called structural stability. It measures what happens when we slightly modify our equation. That is, how do our solutions change if we change from $\dot{x} = f(x) \to \dot{x} = \tilde{f}(x)$, where $f(x)$ is "close" to $\tilde{f}(x)$?

What do we mean by close to? This is not an easy question to answer in general, but for our purposes, we mean that

$$\tilde{f}(x) = f(x) + \varepsilon g(x),$$

where g is a smooth function and $\varepsilon \ll 1$ is a very small parameter. Actually, this is still too complicated for our purposes, so we will restrict ourselves to cases where g is constant (or maybe linear), so

$$\boxed{\tilde{f}(x) = f(x) + \varepsilon,} \quad \text{or maybe} \quad \tilde{f}(x) = f(x) + \varepsilon x.$$

In the case of our population models, ε corresponds to a small amount of immigration (if $\varepsilon > 0$) or emmigration (if $\varepsilon < 0$).

We want to know what happens to our system when we add this small perturbation? For instance, what happens if a few fish swim upstream into our lake every year or if one fisherman takes an extra one or two fish per year. One would hope that the dynamics wouldn't change much.

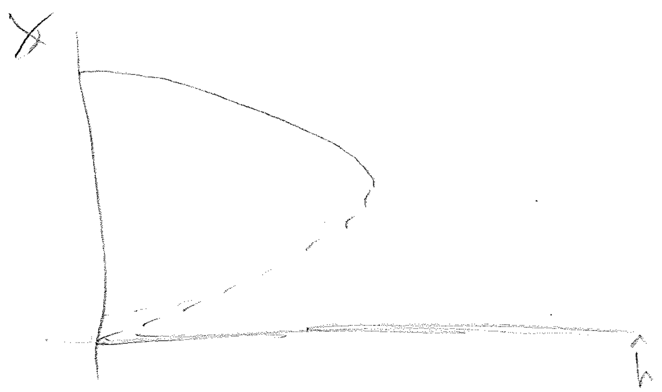
Case 1:

$$\dot{x} = rx(1-x) - h \Rightarrow \dot{x} = rx(1-x) - h + \varepsilon$$

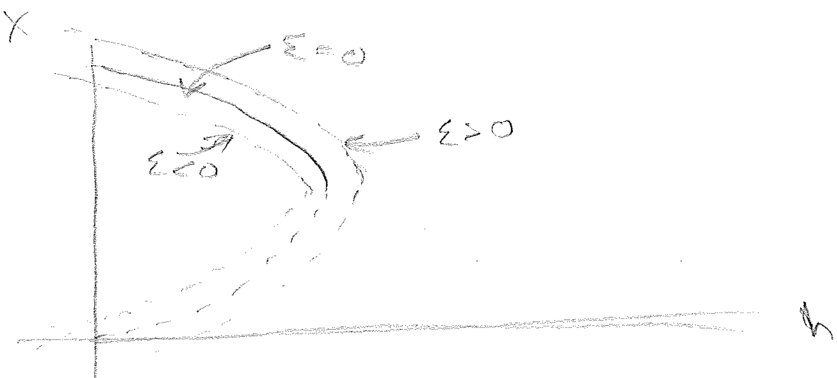
If we let $\hat{h} = h - \varepsilon$, then

$$\dot{x} = rx(1-x) - \hat{h}.$$

This is exactly the same as our original model, just with different names, so we get the same bifurcation diagram:



The only difference is that our h-axis has been shifted slightly. Combining these into one graph, we get



Nothing really changed qualitatively. If there's a little bit of immigration, the stable equilibrium shifts up a bit; if there's a little bit of emmigration, the stable equilibrium shifts down a bit.

Option 2:

Now suppose we're using option 2, so

$$\dot{x} = rx(1-x) - hx.$$

If we perturb our model slightly, we get

$$\dot{x} = rx(1-x) - hx + \varepsilon.$$

Let's do our analysis again. We have

$$\dot{x} = \tilde{f}(x) = rx - rx^2 - hx + \varepsilon,$$ so

the equilibria are where $\tilde{f}(x) = 0$.

$$\tilde{f}(x) = 0 \Rightarrow rx^2 + (h-r)x - \varepsilon = 0$$

$$\Rightarrow x = \frac{r - h \pm \sqrt{(h-r)^2 + 4r\varepsilon}}{2r}$$

There could be 2, 1 or 0 equilibria.

2 eq. If $(h-r)^2 + 4r\varepsilon > 0$

Notice that this is always true if $\varepsilon > 0$ (because $r > 0$ as well). If $\varepsilon < 0$, then

$$(h-r)^2 > -4r\varepsilon$$

$$\Rightarrow h - r > \sqrt{-4r\varepsilon} \quad \text{or} \quad h - r < -\sqrt{-4r\varepsilon}.$$

$$\Rightarrow h > r + \sqrt{-4r\varepsilon} \quad \text{or} \quad h < r - \sqrt{-4r\varepsilon}$$

1 eq. This is only possible if $\varepsilon < 0$. In that case, we get one equilibria when $(h-r)^2 = -4r\varepsilon$, so

$$h - r = \pm\sqrt{-4r\varepsilon} \Rightarrow h = r \pm \sqrt{-4r\varepsilon}$$

0 eq. Again, this is only possible if $\varepsilon < 0$.

In this case, we need

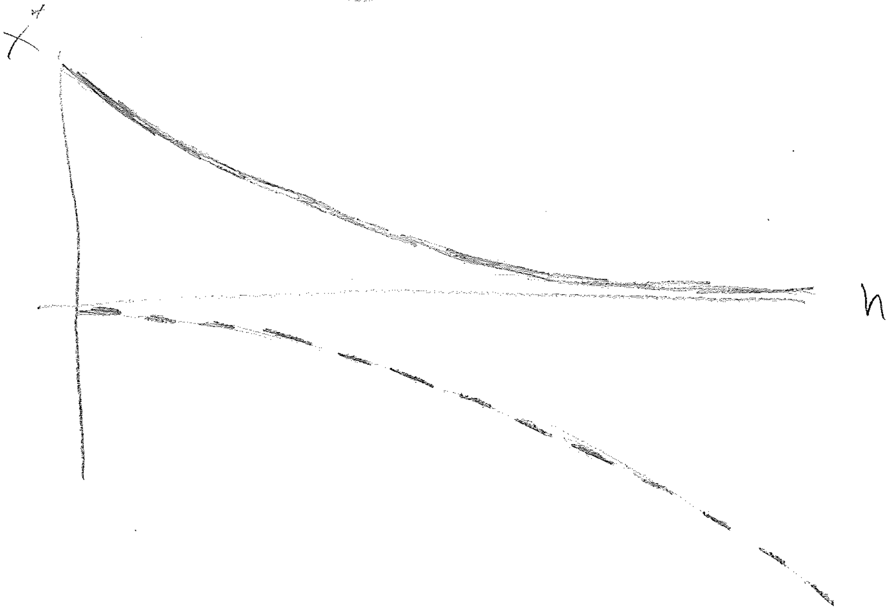
$$(h-r)^2 + 4r\varepsilon < 0$$

$$\Rightarrow (h-r)^2 < -4r\varepsilon$$

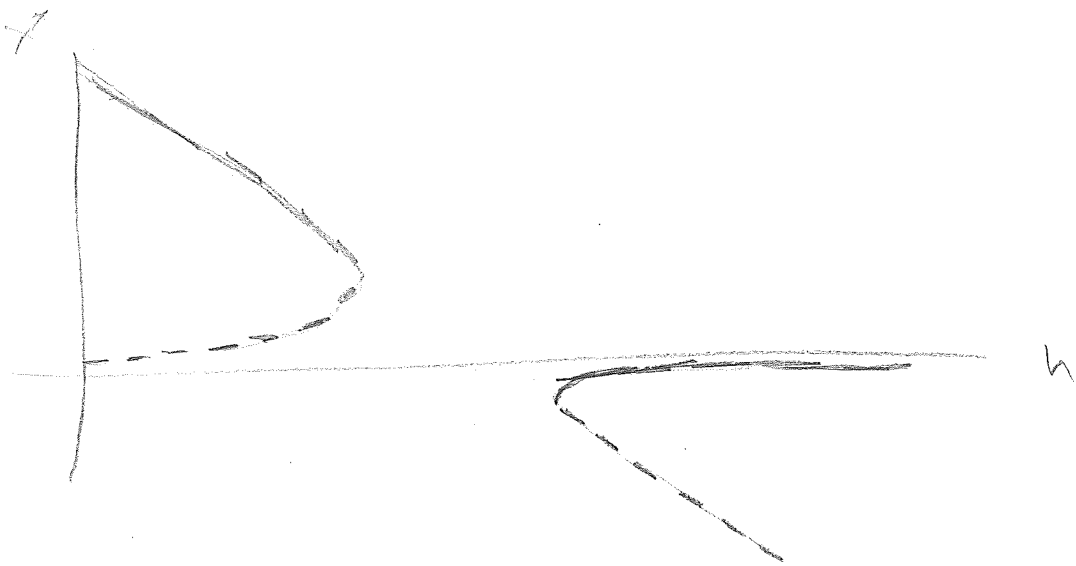
$$\Rightarrow (h-r) < \sqrt{-4r\varepsilon} \quad \text{or} \quad (h-r) > -\sqrt{-4r\varepsilon}$$

$$\Rightarrow h < r + \sqrt{-4r\varepsilon} \quad \text{or} \quad h > r - \sqrt{-4r\varepsilon}$$

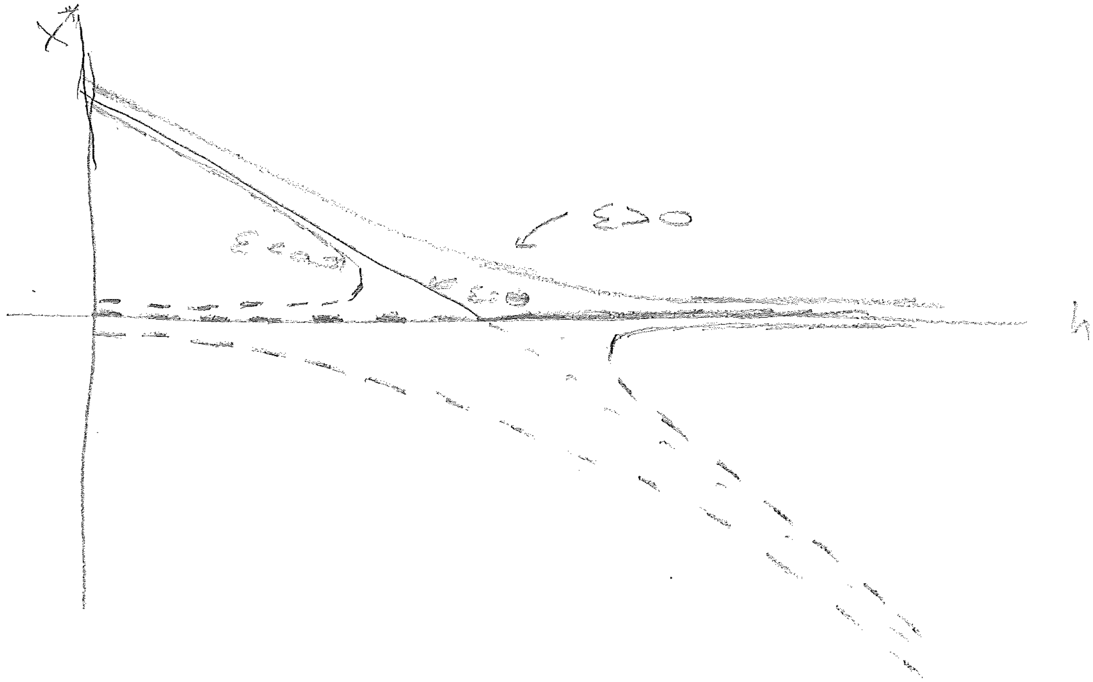
$\varepsilon > 0$ diagram



$\varepsilon < 0$ diagram



Combining these together, we get



This means that the bifurcation diagram changes drastically even when a very small ε-perturbation is applied. In particular, the number of fixed points changes. We call the transcritical bifurcation <u>structurally unstable</u>.

Important Question:

We said that, to maximize yield, we should set $h = \frac{r}{2}$. If ε is very small, then it seems from our graph that the positive equilibrium would still be stable. How big or ε would we need to lose the positive equilibrium at $h = \frac{r}{2}$?

It should be clear that we need $\varepsilon < 0$ for any loss of equilibrium to occur. We know from before that there is exactly one equilibrium when $h = r \pm \sqrt{-4r\varepsilon}$. We're interested in $h = \frac{r}{2}$, so

$$\frac{r}{2} = r \pm \sqrt{-4r\varepsilon}$$

$$\Rightarrow -\frac{r}{2} = \pm\sqrt{-4r\varepsilon}$$

$$\Rightarrow -\frac{r}{2} = -\sqrt{-4r\varepsilon}$$

$$\Rightarrow \frac{r}{2} = \sqrt{4r\varepsilon}$$

$$\Rightarrow \frac{r^2}{4} = 4r\varepsilon$$

$$\Rightarrow \boxed{\frac{r}{16} = \varepsilon}$$

This means that if the emmigration rate is more than $1/16$ of the growth rate, the population would collapse under our proposed plan.

Stability Diagram:

We can summarize these results in the following diagram: