Overview:

Last week we focused on differential equations (continuous time). We've already seen that differential equations arise from discrete time maps when we let the time step go to zero. For instance, we modeled compound interest w/ the equation

$$P(t+\Delta t) = P(t) + r\Delta t P(t) = (1+r\Delta t)P(t).$$

If we choose $\Delta t = 1$, we get annually compounded interest:

1) $P_{t+1} = (1+r)P_t = RP_t$

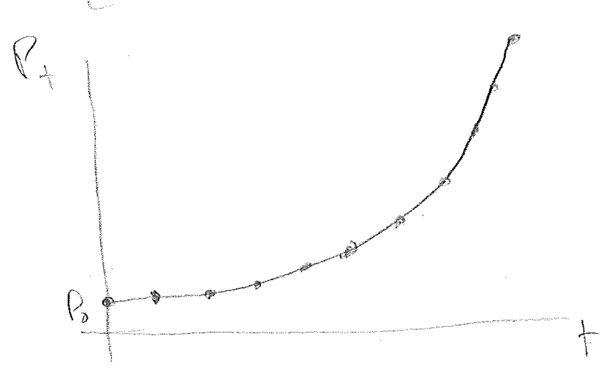
If we let $\Delta t \to 0$, we get continuously compounded interest:

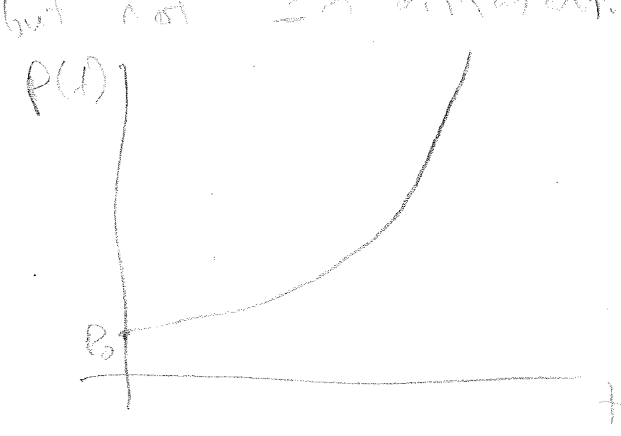
2) $\dfrac{dP}{dt} = rP(t)$

We have already solved these equations. If $P(0) = P_0$, then

$$P_t = P_0 R^t \quad \text{and} \quad P(t) = P_0 e^{rt}.$$

These equations are different, but not so different.



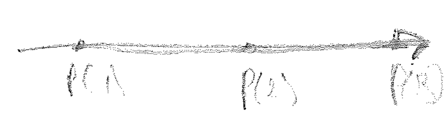


If these models are so similar, why should we choose one over the other?

- Ease of computation – While linear difference equations are relatively easy to solve, nonlinear maps become intractable very quickly (even more so than ODEs).

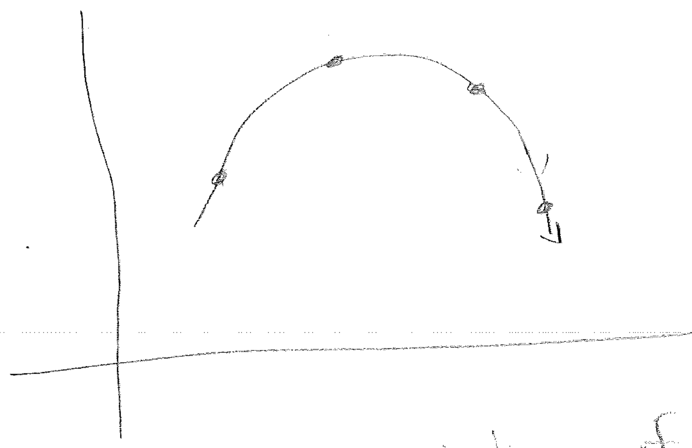
- Realism – Some real-world phenomena really are discrete. Your bank almost certainly compounds interest at discrete steps, and even if they didn't, you get paid at regular intervals; Many species only reproduce at discrete breeding seasons (think salmon or mayflies); etc. In fact, many results from quantum mechanics suggest that essentially everything is discrete.

- They aren't exactly the same, and it can matter a lot more than you might think. The fundamental difference is that discrete maps "hop" from point to point, while continuous functions "flow" through points.



Why is this hopping so important?

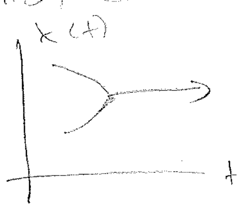
Can this be the solution of an ODE $x'(t)=f(x)$?

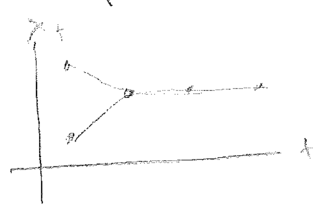


Can this be a solution of a difference equation $x_{t+1}=f(x_t)$?

This is one key difference: solutions of difference equations can turn around!

Another important point is that





solutions can meet!

Before we get too much farther, let's make sure we can solve some simple difference equations. We will do so by analogy with ODEs.

How do you solve the equation $y' = ay$, $y(0) = y_0$?
One common method is to guess the right form, then fill in some constants.

For instance, we can guess $y(t) = e^{rt}$, for some constant r. We then get $y'(t) = re^{rt}$, so

$$re^{rt} = ae^{rt} \Rightarrow r = a.$$

This means that $y(t) = e^{rt}$ is a solution to $y' = ay$, but it doesn't match our initial condition. However, since this is a linear equation, any multiple of y is also a solution, so $y(t) = Ce^{rt}$ for some C. We therefore have $y(0) = C = y_0$, so $\boxed{y(t) = y_0 e^{rt}}$.

We can solve difference equations the same way.

- i.e. $x_{t+1} = a x_t$, $x_0 = x_0$

We will guess a power law: $x_t = \lambda^t$.
$\Rightarrow x_{t+1} = \lambda^{t+1}$

$\lambda^{t+1} = a\lambda^t \Rightarrow \lambda = a$, so $x_t = a^t$ is a solution.

This doesn't match our initial conditions, but this is a linear problem, so any multiple of x_t is also a solution to our problem, so

$x_t = C a^t$ is a solution for any C.

We therefore have

$$x_0 = C a^0 = C = x_0 \Rightarrow \boxed{x_t = x_0 a^t}.$$

Similarly, we can solve inhomogeneous problems. Consider the problem $y'(t) = ay + b$, $y(0) = y_0$.

We can solve this in two steps:

1) pretend $b = 0$ and solve the homogeneous problem $\hat{y}'(t) = a\hat{y} \Rightarrow \hat{y}(t) = Ce^{at}$.

2) Next, guess the form of a particular solution to $y' = ay + b$. Since b is a constant, we'll guess a constant too.

$$\tilde{y}_1 = B \Rightarrow \tilde{y}' = 0 \Rightarrow 0 = aB + b \Rightarrow B = -\frac{b}{a}$$

so $\tilde{y} = -\frac{b}{a}$.

The general solution is the sum of the homogeneous and particular solutions:

$y(t) = \hat{y} + \tilde{y} = Ce^{at} - \frac{b}{a}$. If we plug in our initial condition, we get

$y(0) = y_0 = C - \frac{b}{a} \Rightarrow C = y_0 + \frac{b}{a}$, so

$$\boxed{y(t) = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}}$$

Again, we can solve difference equations in the same manner:

$$x_{t+1} = a x_t + b$$

1) Solve Homogeneous problem

$$x^H_{t+1} = a x^H_t$$

$$\Rightarrow \quad x^H_t = C a^t$$

2) Guess form for particular solution:

$$x^P_t = B \quad \Rightarrow \quad x^P_{t+1} = B$$

$$\Rightarrow \quad B = aB + b$$

$$\Rightarrow \quad B = \frac{b}{1-a}$$

Full solution is sum of homogeons and particular solutions:

$$x_t = x^H_t + x^P_t = C a^t + \frac{b}{1-a}$$

Our initial condition means that

$$x_0 = C a^0 + \frac{b}{1-a} = C + \frac{b}{1-a}$$

$$\Rightarrow \quad C = x_0 - \frac{b}{1-a}$$

$$\Rightarrow \quad \boxed{x_t = \left(x_0 - \frac{b}{1-a}\right) a^t + \frac{b}{1-a}}$$

Note that the ode $y' = ay$ has an equilibrium at $y^* = 0$, because if $y = 0$, then $y' = 0$, so the solution never changes. Likewise, the difference equation $x_{t+1} = a x_t$ has an equilibrium at $x^* = 0$, because if $x_t = 0$ then $x_{t+1} = 0$, so the solution never changes. In general, an ODE $x' = f(x)$ has an equilibrium at x^* iff $f(x^*) = 0$. Likewise, a DE has an equilibrium at x^* iff $f(x^*) = x^*$.

We can also talk about stability of fixed points. For comparison, if $x'(t) = f(x)$ has an equilibrium at x^*, then we find its stability by linearizing about x^*. That is, we make a new variable $y(t) = x(t) - x^*$, so

$y' = x'$ and

$$y'(t) = f(y + x^*) \approx f(x^*) + f'(x^*)y + \underbrace{\frac{f''(x^*)}{2} y^2 + \ldots}_{\text{y is small, so these are very small}}$$

$$\Rightarrow y' \approx \cancelto{0}{f(x^*)} + f'(x^*) y$$

$$\Rightarrow y' \approx f'(x^*) y.$$

This is a linear problem – we know how to solve it.

In particular, $y(t)$ grows if $f'(x^*)>0$, which means $x(t)-x^*$ grows, so x^* is unstable. Likewise, $y(t)$ shrinks if $f'(x^*)<0$, which means $x(t)-x^*$ shrinks, so x^* is stable.

Similarly, If $x_{t+1}=f(x_t)$ has an equilibrium at x^*, then we will let $y_t=x_t-x^*$, so $y_{t+1}=x_{t+1}-x^*$, so

$$y_{t+1}+x^* = f(y_t+x^*) \approx f(x^*)+f'(x^*)y_t \underbrace{+\frac{f''(x^*)}{2}y_t^2+\dots}_{\substack{y_t \text{ is small, so} \\ \text{those are very} \\ \text{small}}}$$

$$\Rightarrow y_{t+1}+\cancel{x^*} \approx \cancel{x^*}+f'(x^*)y_t$$

$$\Rightarrow y_{t+1}\approx f'(x^*)y_t.$$

This is a linear problem — we know how to solve it. In particular, if $|f'(x)|<1$, then y_t shrinks, so x_t-x^* shrinks, so x^* is stable. If $|f'(x)|>1$, then y_t grows, so x_t-x^* grows, so x^* is unstable.