AMATH 383 – Homework 1

Lowell Thompson

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Problem 1

A professor's daughter just started her freshman year at a private college that costs \$30,000 per year. When she was born (18 years ago) her grandparents put \$10,000 in a college fund for her. If the interest were compounded continuously, what annual percentage rate (APR) would the college fund need so that she has enough money today for the next four years of college tuition? What if the interest were compounded monthly? What if it were compounded yearly?

Solution

Recall from class that the value of an investment with continuously compounded interest is given by

$$P(t) = P_0 e^{rt}, \tag{1}$$

where P_0 is the initial value and r is the APR. We know that $P_0 = 10,000$ and $t = 18$ and we want $P(18)$ to equal 120,000. This means

$$\begin{aligned} P(18) = 10,000e^{18r} &= 120,000, \text{ so} \\ e^{18r} &= 12, \text{ and} \\ 18r &= \ln 12, \text{ and therefore} \\ r &= \frac{1}{18}\ln 12. \end{aligned}$$

This means that the fund would need an APR of approximately 13.81%.

If the interest were compounded monthly, then (1) would become

$$P(t) = P_0 \left(1 + \frac{r}{12}\right)^{12t}, \tag{2}$$

so we have

$$
\begin{aligned}
P(18) = 10,000\left(1+\frac{r}{12}\right)^{12\cdot 18} &= 120,000, \text{ so} \\
\left(1+\frac{r}{12}\right)^{216} &= 12, \text{ so} \\
1+\frac{r}{12} &= 12^{1/216}, \text{ so} \\
r = 12\cdot\left(12^{1/216}-1\right).
\end{aligned}
$$

This means that the fund would need an APR of approximately 13.88%.

Finally, if the interest were compounded yearly, then (1) would become

$$P(t) = P_0(1+r)^t, \tag{3}$$

so we have

$$
\begin{aligned}
P(18) = 10,000(1+r)^{18} &= 120,000, \text{ so} \\
(1+r)^18 &= 12, \text{ so} \\
1+r &= 12^{1/18}, \text{ and} \\
r &= 12^{1/18}-1.
\end{aligned}
$$

This means that the fund would need an APR of approximately 14.80%.

Problem 2

You have just signed a contract that entitles you to receive \$1,000,000 twenty years from now, but you can't wait and want your money now. Assuming that the risk-free, inflation adjusted APR is 3% per year, compounded continuously, what is a fair price for your contract?

Solution

The future value of our contract is governed by

$$P(t) = P_0 e^{rt}, \tag{4}$$

where r is the inflation rate and P_0 is the present value of our contract. We know that the value of the contract in 20 years will be $1,000,000 and that $r = 0.03$. We therefore have

$$\begin{aligned} P(18) = P_0 e^{0.03\cdot 20} &= 1,000,000, \text{ so} \\ P_0 e^{0.6} &= 1,000,000, \text{ and} \\ P_0 &= 1,000,000 e^{-0.6}. \end{aligned}$$

This means that our contract is currently worth $P_0 \approx \$550,000$.

Problem 3

Let N_t be the number of rabbits in Australia at the end of year t and let M_t be the number of fox in Australia at the end of year t. Suppose that the populations interact in such a way that

$$\begin{aligned} N_{t+1} &= 6N_t - 4M_t, \\ M_{t+1} &= 2N_t. \end{aligned} \tag{5}$$

If $N_0 = M_0 = 1$, find the populations N_t and M_t for all $t \geq 0$.

Extra credit (5 points): There is a very easy way to solve the preceding problem, but it only works because we chose convenient initial conditions. A slightly more systematic way to solve the problem is by assuming N_t and M_t are of the form

$$\begin{pmatrix} N_t \\ M_t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \lambda^t \tag{6}$$

and determining the values of λ, u and v. Find a condition on N_0 and M_0 such that the solution will be of this form, and solve (**??**) for any such initial conditions. Show that this agrees with your solution from above.

Solution

First, note that if $N_0 = M_0 = 1$, then $N_1 = 6N_0 - 4M_0 = 2$ and $M_1 = 2N_0 = 2$. We then find that $N_2 = 6N_1 - 4M_1 = 4$ and $M_2 = 2N_1 = 4$. Similarly, $N_3 = 6N_2 - 4M_2 = 8$ and $M_3 = 2N_2 = 8$. It should be clear that $N_t = M_t = 2^t$. To check that this method works for all t, we just need to see that if we've proved it for the first n steps, then we can prove one more step. To this end, suppose that we know that $N_n = M_n = 2^n$ for some whole number n. We therefore have

$$N_{n+1} = 6N_n - 4M_n = 6 \cdot 2^n - 4 \cdot 2^n = 2 \cdot 2^n = 2^{n+1}. \tag{7}$$

Likewise,

$$M_{n+1} = 2N_n = 2 \cdot 2^n = 2^{n+1}. \tag{8}$$

This means that $N_t = M_t = 2^t$.

For the extra credit, suppose that $N_t = u\lambda^t$ and $M_t = v\lambda^t$ for some constants u, v and λ. In particular, this means that $u = N_0$ and $v = M_0$. From (5), we know that

$$\begin{aligned} N_{t+1} &= 6N_t - 4M_t = 6N_0\lambda^t - 4M_0\lambda^t = (6N_0 - 4M_0)\lambda^t, \text{ and} \\ M_{t+1} &= 2N_t = 2N_0\lambda^t. \end{aligned} \tag{9}$$

From our assumption, we also know that $N_{t+1} = u\lambda^{t+1} = N_0\lambda^{t+1}$ and $M_{t+1} = v\lambda^{t+1} = M_0\lambda^{t+1}$. We therefore have

$$\begin{aligned} N_0\lambda^{t+1} &= (6N_0 - 4M_0)\lambda^t, \text{ and} \\ M_0\lambda^{t+1} &= 2N_0\lambda^t. \end{aligned} \tag{10}$$

Solving each of these equations for λ and setting them equal to each other, we find that

$$6 - 4\frac{M_0}{N_0} = 2\frac{N_0}{M_0}. \tag{11}$$

After some algebra, this becomes

$$(N_0 - 2M_0)(N_0 - M_0) = 0, \tag{12}$$

so it must be true that either $N_0 = M_0$ or $N_0 = 2M_0$. If $N_0 = M_0$, then (10) tells us that $\lambda^{t+1} = 2\lambda^t$, so $\lambda = 2$. If $N_0 = 2M_0$, then (10) tells us that $\lambda^{t+1} = 4\lambda^t$, so $\lambda = 4$. We therefore have two possible solutions, depending on our initial conditions.

If $N_0 = M_0$, then the solution is

$$\begin{pmatrix} N_t \\ M_t \end{pmatrix} = \begin{pmatrix} N_0 \\ M_0 \end{pmatrix} 2^t, \tag{13}$$

and if $N_0 = 2M_0$, then the solution is

$$\begin{pmatrix} N_t \\ M_t \end{pmatrix} = \begin{pmatrix} N_0 \\ M_0 \end{pmatrix} 4^t. \tag{14}$$

If $N_0 = M_0 = 1$, then we are in the first case, and this solution matches our previous answer.

Problem 4

In a growing organism, metabolism supplies energy to both maintain existing tissues and create new tissues by cell division. Let Y_c be the metabolic rate of a single cell (that is, the amount of energy used by the cell per unit time) and let E_c be the energy required to create a new cell and let $N_c(t)$ be the total number of cells at time t. The total mass of the organism is given by

$$m(t) = m_c N_c(t), \tag{15}$$

and the total metabolic rate Y of the organism is given by

$$Y(t) = Y_c N_c(t) + E_c \frac{\mathrm{d}N_c}{\mathrm{d}t}. \tag{16}$$

It has been argued that Y is related to m by the equation

$$Y(t) = Y_0 \cdot (m(t))^{3/4}, \tag{17}$$

where Y_0 is some constant.

(a) Show that equations (15), (16) and (17) can be combined to obtain

$$\frac{\mathrm{d}m}{\mathrm{d}t} = am^{3/4} - bm, \tag{18}$$

where $a \equiv Y_0 m_c / E_c$ and $b = Y_c/E_c$ are constants.

(b) Suppose that when an organism matures, its mass stops changing. That is, at maturity $m(t) \equiv M$ is constant, so $m'(t) = 0$. Find M, and show that (18) can be rewritten as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = am^{3/4}\left[1 - \left(\frac{m}{M}\right)^{1/4}\right]. \tag{19}$$

(c) Let $r = (m/M)^{1/4}$ and $R = 1 - r$. Show that (19) becomes

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\left(\frac{a}{4M^{1/4}}\right) R. \tag{20}$$

Solve this differential equation for $R(t)$ with initial condition $R(0) = R_0$ and plot $\ln(R(t)/R_0)$ vs $\tau \equiv at/(4M^{1/4})$. This plot should be a straight line with slope of -1, regardless of the values of any constants.

Part (a)

From (15), we know that $N_c(t) = m(t)/m_c$, so $N_c'(t) = m'(t)/m_c$. Substituting these into (16), we find that

$$Y(t) = \frac{Y_c}{m_c} m(t) + \frac{E_c}{m_c} \frac{\mathrm{d}m}{\mathrm{d}t}. \tag{21}$$

Substituting (17) into this, we find that

$$\begin{aligned} Y_0 m^{3/4} &= \frac{Y_c}{m_c} m + \frac{E_c}{m_c} \frac{\mathrm{d}m}{\mathrm{d}t}, \text{ so} \\ \frac{E_c}{m_c} \frac{\mathrm{d}m}{\mathrm{d}t} &= Y_0 m^{3/4} - \frac{Y_c}{m_c} m, \text{ and} \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= \frac{Y_0 m_c}{E_c} m^{3/4} - \frac{Y_c}{E_c} m. \end{aligned}$$

If we let $a = Y_0 m_c / E_c$ and $b = Y_c / E_c$, then this means that

$$\frac{\mathrm{d}m}{\mathrm{d}t} = a m^{3/4} - b m, \tag{22}$$

as desired.

Part (b)

To find M, we set (18) equal to zero, so

$$\begin{aligned} 0 &= a M^{3/4} - b M, \text{ and} \\ b M &= a M^{3/4}, \text{ so} \\ M^{1/4} &= \frac{a}{b}, \text{ so} \\ M &= \left(\frac{a}{b}\right)^4. \end{aligned}$$

Substituting this into (18), we obtain

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &= am^{3/4} - bm \\ &= am^{3/4}\left[1 - \frac{bm}{am^{3/4}}\right] \\ &= am^{3/4}\left[1 - \left(\frac{b}{a}\right)m^{1/4}\right] \\ &= am^{3/4}\left[1 - \left(\frac{1}{M^{1/4}}\right)m^{1/4}\right] \\ &= am^{3/4}\left[1 - \left(\frac{m}{M}\right)^{1/4}\right],\end{aligned}$$

as desired.

Part (c)

Now let $r = (m/M)^{1/4}$ and $R = 1 - r$. Using the chain rule, we find that

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= -\frac{1}{4}\left(\frac{m}{M}\right)^{-3/4} \cdot \frac{1}{M} \cdot \frac{\mathrm{d}m}{\mathrm{d}t} \\ &= -\frac{1}{4M^{1/4}} \cdot m^{-3/4} \cdot am^{3/4}\left[1 - \left(\frac{m}{M}\right)^{1/4}\right] \\ &= -\frac{a}{4M^{1/4}}R,\end{aligned}$$

as desired.

This equation is separable, and we find that

$$R(t) = R_0 e^{-at/(4M^{1/4})}, \tag{23}$$

so

$$\ln\frac{R(t)}{R_0} = -\frac{at}{4M^{1/4}} \equiv -\tau. \tag{24}$$

Therefore, the plot of $\ln(R(t)/R_0)$ vs τ is a straight line through the origin with slope -1.