AMATH 383 – Suggested Term Paper Topics

This is a list of potential topics for your term paper. It is by no means comprehensive, and you are welcome to propose your own topic if you find one that interests you. If you do choose a topic that is not from this list, you should run it by me first just to make sure that it fits the framework of the assignment.

There are three general types of papers you can choose:

1. You can find several (probably 3-4) scientific papers discussing models of a common theme. Your paper should explain these papers in light of the three phases of modeling that we discussed in class: Formulation, Solution and Discussion. You should describe how and why these models were formulated in the way that they were (including why the different papers proposed different models), then explain the mathematical analysis in detail and discuss how the results compare with the real-world phenomena being modeled. You will find that research papers usually skimp on the mathematical description, so you'll need to fill in many of the details yourself. Most of the neural models described below are in this category.

2. For older, more established topics, such as many problems in classical mechanics, there are not many research papers available. In these cases, you might need to use textbooks for the bulk of your references. A paper on one of these problems will focus less on the formulation (since all researchers tend to agree on the right way to describe the problem) and more on analysis and discussion. In these cases, you should spend more time explaining the mathematical analysis and the implications of these models for real world problems.

3. Some problems lend themselves to more of an engineering approach. For instance, one of the suggested topics in neural networks involves implementing your own network and using it to solve a practical problem. In this case, you will still need to write a paper describing why you chose the details of your network, why Hopfield networks (or whichever version you choose to implement) are a good choice for the problem, what pitfalls the approach may have and how your implementation faired. You should also give at least some mathematical analysis of the method. However, if your project involves a substantial amount of implementation then you will need less in the full paper. (If you decide on this type of topic, you should discuss this work balance in your proposal.)

Note that you are *not* expected to create a novel research project or an entirely new model. You are also *not* expected to collect your own data for any of these projects.

Neural Models

Neuroscience is a very active area of research in mathematical biology. The field has tended to focus on dynamical systems approaches, although stochastic models are also gaining popularity. The wide variety of spatial scales in neuroscience leads to an equally wide variety of mathematical models: everything from models of ion flow through individual cell-membrane channels to the dynamics of large networks of neurons. On the smaller end of this range, one of the most popular models of neuron spiking is the Hodgkin-Huxley model. A nice introduction can be found here: `http://www.bem.fi/book/04/04.htm`, and the original paper is *A quantitative description of membrane current and its application to conduction and excitation in nerve* by A.L. Hodgkin and A.F. Huxley, 1952.

An interesting topic would be to start with the Hodgkin-Huxley model and investigate more complex modifications of the theory. For example, one could look at what happens when two or more cells are coupled together and each is governed by Hodgkin-Huxley equations. A good place to start would be *Alpha-frequency rhythms desynchronize over long cortical distances: A modeling study* by S.R. Jones, D. Pinto, T. Kaper and N. Kopell, 2000.

Another interesting avenue would be to investigate simplified models that capture most of the interesting behavior of the Hodgkin-Huxley. A good place to start would be *Bifurcation analysis of a two-dimensional simplified Hodgkin-Huxley model exposed to external electric fields* by H. Wang, Y. Yu, S. Wang and J. Yu, 2014.

You could also look at how to incorporate stochasticity into the Hodgkin-Huxley model. Since voltage spikes in neural membranes are based on the movement of ions through small channels, random changes in ion density or channel activity can have important effects on neuron activity. A good place to start would be *The what and where of adding channel noise to the Hodgkin-Huxley equations* by J.H. Goldwyn and E. Shea-Brown, 2011.

Another interesting project would be to look at models of neural behavior in different organisms. For many species, the Hodgkin-Huxley theory doesn't seem to be appropriate, so people tend to use other models such as simple oscillators. You could work through some of these models, explain why the differences are needed (from a physiological point of view) and how the behavior of these models differs. Two good places to start are *Neural mechanism of optimal limb coordination in crustacean swimming* by C. Zhang et al., 2014 and *Potential, impedance, and rectification in membranes* by D.E. Goldman, 1943.

On a larger scale, people generally ignore variations in voltage and just think of a neuron as firing or not firing. In this case, the most important variable is usually the firing frequency. The classic model (and still one of the most popular) is the Hopfield network model. The original papers are *Neural computation of decisions in optimization problems* by J.J. Hopfield and D.W. Tank, 1985 and *Neural networks and physical systems with emergent collective computational abilities* by J.J. Hopfield, 1982.

In these network models, most of the behavior is governed by the connection matrix T_{ij}, which describes the connections between different neurons. Using the Hopfield papers as a starting point, you could discuss how the qualities of this matrix (e.g., sparsity) influence the behavior of the model. A particularly interesting direction might be to look at what happens if one does not use a symmetric matrix.

If you are interested in more of a computer science / engineering project, you could construct your own artificial Hopfield network, train it on a suitable dataset (there are many freely available online; you don't need to collect your own data) and use it for pattern recognition. For instance, you could make a Hopfield network that recognized hand writing. A good example can be found at `http://web.cs.ucla.edu/~rosen/161/notes/hopfield.html`. For a project like this, you would still need to do a write up explaining how Hopfield networks work and why they are an appropriate choice for whatever problem you choose to work on. As a ballpark estimate, roughly half of your work should be on the paper and another half on implementing your network.

Ecological Models

We have talked about (or will talk about) several models in population ecology (one species) and community ecology (several species). For example, we discussed logistic growth; we had a homework problem on spruce budworm dynamics; and soon we will discuss the Lotka-Volterra model for predator-prey interactions. These models are excellent starting points for many projects.

For example, you could start with a basic model such as Lotka-Volterra and incorporate more biologically realistic terms and analyze the resulting models. Any discussion of this model should probably incorporate notions of structural stability, and it would also be interesting to show how the equations can be modified to make parameters more easily measured. (The carrying capacities, for instance, are often very difficult to measure in the field.) A good starting point might be *Dynamical Behaviors determined by the Lyapunov function in competitive Lotka-Volterra systems* by T. Ying, R. Yuan and Y. Ma, 2013.

Another interesting idea would be to investigate stochastic versions of the Lotka-Volterra model. We will look at stochastic versions of the logistic equation soon, and similar techniques could be applied to predator-prey models or other interacting populations. The Lotka-Volterra model is a particularly interesting candidate because it is structurally unstable, so stochasticity tends to have a large effect. A good place to start would be *The Helmholtz theorem for the Lotka-Volterra equation, the extended conservation relation, and stochastic predator-prey dynamics* by Y.-A. Ma and H. Qian, 2014.

Mechanical Models

Although the basic laws of classical mechanics have been well understood for hundreds of years, even moderately complex problems tend to have very interesting dynamics. Many of these problems are old enough that journal articles on the subjects are either nonexistent or difficult to comprehend (try reading Newton's Principia Mathematica if you don't believe me). For these problems, you may want to consult textbooks rather than recent papers. In addition, the steps of model formulation are often somewhat different than what we have discussed – the formulation tends to be very easy because everyone agrees on Newton's laws, but the analysis and conclusions can be much more difficult.

The so-called two-body problem is to use the Newtonian theory of gravity to predict the motion of two interacting objects, such as the sun and the earth or a space shuttle and the earth. This problem has been completely solved, and is fairly easy to analyze. Somewhat surprisingly, once one more body is introduced (for instance, if we want to model the sun, earth and moon), the equations of motion (i.e., the system of differential equations) do not become much more complicated, but the problem becomes far more difficult. As recently as 2013, physicists have discovered new classes of solutions to this problem. You could write a paper working out some of the special solutions to the three-body problem and discussing why it is so much more complicated than with two objects.

Another classical mechanics problem with surprisingly rich dynamics is the double pendulum. (A good introduction can be found here: `https://www.myphysicslab.com/pendulum/double-pendulum-en.html`). Despite the fact that this system is fairly low-dimensional and conserves energy, it still exhibits chaotic dynamics. You could analyze models of the single pendulum (quite easy) and the double pendulum (more complicated), find trajectories of this system along constant energy surfaces and explain why one system exhibits chaos while the other does not.

If you are particularly motivated, you could also investigate the problem of controlling the double pendulum so that it reaches a specified position. When you can control the position and velocity of both pendulums, this is quite easy, but a real system usually can only control one of the pendulums. You could analyze various control schemes and discuss the differences (both practical and mathematical) between them.

Chemical Kinetics

One particularly interesting problem in chemical kinetics is that of chemical oscillations. For example, *Experimental modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction* by I. Lengyel, G. Rabai and I.R. Epstein, 1990 describes a model of chemical reactions that exhibit large and regular oscillations in concentrations. This phenomenon was long thought to be impossible. If you investigate the model carefully, you will find that the equilibrium is not actually a spiral, which means that the oscillations are actually a transient phenomenon. You could then analyze these oscillations using, among other possibilities, the auto-correlation function of the oscillatory trajectories.

Molecular Biology

Modeling the folding of proteins is a highly active area of research in biology and biophysics. One particularly interesting method is known as the Markov state method. An excellent introduction can be found at `https://www.ncbi.nlm.nih.gov/pmc/articles/PMC2933958/`. In this method, one chooses a set of "microstates" that can describe the shape of a protein molecule and then describes how the protein randomly moves between these states. You could analyze some of these models and discuss the difficulties involved in formulating a model simple enough to analyze properly but rich enough to capture the behavior of important protein dynamics.

Another interesting topic in molecular biology is that of the molecular motor. This term describes a molecule or small group of molecules capable of moving consistently in a controllable direction (with a continuous input of energy). Such motors occur naturally in roles such as muscle contraction and DNA repair, and over the last twenty years there have been many proposals for synthetic molecular motors. Many models have been proposed for many different types of motors, and most of these models have an interesting blend of mechanics and stochastic processes. A good place to start might be *Molecular motors* by M. Schliwa and G. Woehlke, 2003.

Population Genetics

There are a wide variety of models describing evolution, changes in allele frequency, and the rise and fall of mutations in populations. We will briefly discuss one of the simplest models of population genetics – the Wright-Fisher model – which was originally described in *Evolution in Mendelian populations* by S. Wright, 1930. Since then, many modifications have been proposed to take into account things like mutation, migration and selection. You could use Wright-Fisher as a starting point and discuss some of the newer and more complex methods.